Asymptotic Performance Analysis and Successive Selected Mapping for PAR Reduction in OFDM

Christian Siegl and Robert F.H. Fischer

Lehrstuhl für Informationsübertragung, Friedrich–Alexander–Universität Erlangen–Nürnberg
Cauerstrasse 7/LIT, 91058 Erlangen, Germany, Email: {siegl,fischer}@LNT.de

Abstract

One major drawback of orthogonal frequency-division multiplexing (OFDM) systems is the high peak-to-average power ratio (PAR) of the transmit signal. In order to overcome this problem, selected mapping (SLM) is a well known method for PAR reduction. In this paper, asymptotic performance analysis of SLM is carried out and a bound on the maximum tolerable PAR values is derived. Starting with the analysis of Gaussian signalling at Nyquist rate, it is also shown that these theoretic results can be applied for QAM signalling at any oversampling rate.

Based on the idea to distinguish between tolerable and non-tolerable PAR values a novel variant of SLM is derived, namely successive SLM (SSLM). With this approach it is possible to significantly decrease the average number of assessed signal candidates and the average number of required side-information bits per OFDM frame as long as the threshold of tolerable PAR values is larger than the critical PAR ($\log(D)$ for Nyquist sampled signals). As a rule of thumb, less than 2.71 (Euler's number) candidates and less than 2.57 bits of side-information are required on average.

This work was supported by Deutsche Forschungsgemeinschaft (DFG) within the framework TakeOFDM under grant FI 982/1-1

I. INTRODUCTION

Over the last years, orthogonal frequency-division multiplexing (OFDM) has become one of the most important modulation schemes for transmission over frequency-selective channels. One of the most serious drawbacks of OFDM is the high peak-to-average power ratio (PAR) of the time-domain transmit signal. Non-linear power amplifiers at the transmitter front-end cause signal clipping, which, in turn, leads to signal distortion and out-of-band radiation. In order to avoid violation of spectral masks by out-of-band radiation, operation at large input power back-off is required. To decrease the input power back-off and therefore increase power efficiency, a transmitter-sided preprocessing, to reduce the transmit signal's peak-power, is indispensable.

In literature, a variety of different PAR reduction schemes has been proposed. The most popular schemes are based on multiple signal representation (*selected mapping (SLM)* [3], [27] or partial transmit sequences [26]), clipping and filtering [2], tone reservation [33], active constellation extension [23], algorithms based on lattice decoding [10], [17], or PAR reduction based on redundant coding techniques [31], [32], [13], [12]. For an overview of different PAR reduction schemes see [18].

All these schemes have in common, that they introduce additional complexity (processing of the original transmit signal) and/or redundancy (coding redundancy or side information). Moreover, some schemes increase average transmit power rather than decreasing peak power. A desirable PAR reduction scheme should significantly reduce the peak-power but require only little additional complexity and redundancy.

This paper analyzes the asymptotic behaviour of SLM and derives a practical bound of maximum tolerable peak-power using SLM. Based on this bound, a novel PAR reduction scheme, namely *successive SLM (SSLM)*, is introduced. This technique is able to significantly decrease both, complexity and redundancy by exhibiting comparable performance as original SLM, if the tolerable peak-power is chosen above the bound. All these considerations are confirmed by numerical simulations.

Recently, many variants of the original SLM approach (see e.g., [8], [24], [21], [11], [19], [15], and the references therein) have been proposed, which reduce the computational complexity or address the issue of transmitting the extra information required in SLM. As SSLM is a generic extension on the original approach of SLM, these variants may straightforwardly be combined with SSLM as well.

This paper is organized as follows: Section II defines the OFDM system model; Section III gives a short review of SLM and derives a practical bound of maximum occurring peak-power. These derivations are given for Gaussian time-domain samples at Nyquist rate as well as QAM signalling with arbitrary oversampling factors. The novel PAR reduction scheme SSLM is introduced in Section IV, and Section V draws some conclusions.

II. OFDM System Model

Subsequently, an OFDM system with D subcarriers is considered. The frequency-domain OFDM symbol is denoted by the vector[1] $\boldsymbol{A} = [A_d]$, $d = 0, \ldots, D-1$, of length D. Each element A_d is i.i.d.ly drawn from an arbitrary M-ary QAM constellation. The frequency-domain OFDM symbol is transformed via the *inverse discrete Fourier transform (IDFT)* into time domain. In order to obtain a good approximate of the continuous-time transmit signal, L-times oversampling is applied. The time-domain OFDM symbol is hence given by $\boldsymbol{a} = \mathrm{IDFT}\{\boldsymbol{A}\}$ with $\boldsymbol{a} = [a_k]$, $k = 0, \ldots, LD-1$, and $a_k = 1/\sqrt{D} \sum_{d=0}^{D-1} A_d \mathrm{e}^{\mathrm{j}2\pi kd/(LD)}$.

Due to the superposition of D subcarriers, the time-domain OFDM signal exhibits a high peak-to-average power ratio

$$\xi \stackrel{\text{def}}{=} \frac{\max_{k=0,\ldots,LD-1} |a_k|^2}{\mathrm{E}\{|a_k|^2\}} \; . \tag{1}$$

As common in literature, we consider as performance measure the *complementary cumulative distribution function (ccdf)*, i.e., the probability that the PAR ξ of an OFDM frame exhibits a certain threshold ξ_{th}:

$$C_{\mathrm{o}}(\xi_{\mathrm{th}}) \stackrel{\text{def}}{=} \Pr\{\xi > \xi_{\mathrm{th}}\} \; . \tag{2}$$

III. Analysis of Asymptotic Behaviour of Selected Mapping

A. Review of Selected Mapping

The fundamental idea of SLM [3], [27] is as follows: given the original frequency-domain OFDM frame $\boldsymbol{A}$, U different (statistically) independent signal candidates (which usually have the same energy as the original one) are generated. These different signal representations may be obtained by U unique, bijective mappings $\mathcal{M}^{(u)}$, $u = 1, \ldots, U$. In the original approach to SLM [3] these mappings have been realized by phase rotations, i.e., an element-wise multiplication of $\boldsymbol{A}$ with phase-vectors $\boldsymbol{p}^{(u)} = [p_d^{(u)}]$, with i.i.d.ly randomly drawn equally distributed elements $p_d^{(u)} \in \{\pm 1, \pm \mathrm{j}\}$, $d = 0, \ldots, D-1$, $u = 1, \ldots, U$. Restricting the phase rotations to multiples of $\pi/2$ does not affect the receiver-sided synchronization algorithm as all signal candidates originate from the same QAM grid as the original signal does. Moreover, this choice of phase-vectors ensures that all U signal candidates are pairwise

[1]Notation: Throughout this paper vectors are denoted as bold letters. Frequency-domain symbols are written as capital letters, time-domain symbols as lower-case letters. The expected value and the variance of a random variable are denoted as $\mathrm{E}\{\cdot\}$ and $\mathrm{Var}\{\cdot\}$. We distinguish between logarithm functions with respect to three different bases: the natural logarithm $\log(x) \stackrel{\text{def}}{=} \log_{\mathrm{e}}(x)$, whereby e denotes Euler's number, and the logarithms to bases 10 and 2, $\log_{10}(x)$ and $\log_2(x)$. Furthermore, we define the binary entropy function as $\mathrm{e}_2(x) \stackrel{\text{def}}{=} -x\log_2(x) - (1-x)\log_2(1-x)$, and $\lceil \cdot \rceil$ specifies rounding to the next integer towards infinity. The Digamma function [36] is termed $\Psi(x)$ and Euler's constant by $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right) = 0.5772\ldots$.

statistically independent. Thus, and according to [39], an optimization of these phase-vectors is not necessary here. Other mappings are possible, too, as, e.g., scrambling of the binary data sequence [7] or permutation of the frequency-domain OFDM frame [20].

All U resulting candidates are transformed into time-domain $\boldsymbol{a}^{(u)} = \mathrm{IDFT}\{\boldsymbol{A}^{(u)}\}$. Out of these that one exhibiting the lowest PAR is chosen for transmission. Computational complexity of SLM is hence mainly governed by assessing the U signal candidates[2] (calculation of IDFTs and evaluating the PARs).

In order to recover the information at the receiver, the transmission of side-information, i.e., the index of the applied mapping is necessary. An uncoded representation of this index requires $\lceil \log_2(U) \rceil$ bits per OFDM frame of redundancy.

Starting from (2), the ccdf of PAR of SLM can easily be calculated and reads [3]

$$C_{\mathrm{s}}(\xi_{\mathrm{th}}) \overset{\text{def}}{=} (C_{\mathrm{o}}(\xi_{\mathrm{th}}))^{U} \ . \tag{3}$$

B. Gaussian Signalling and Asymptotic Results

Subsequently, we analyze SLM assuming Gaussian time-domain samples. This assumption, although not exactly met in practical schemes, proves to be very accurate and useful for understanding the behavior of PAR reduction schemes. Basically, this is similar to asymptotic results in information theory where Gaussian signalling is studied. In the present case, i.i.d. Gaussian frequency-domain symbols A_d are assumed leading directly to i.i.d. Gaussian time-domain symbols a_k after the IDFT.

For i.i.d. Gaussian symbols at Nyquist rate, the ccdf of the original OFDM frames can be given analytically, and it reads [3]

$$C_{\mathrm{o}}^{\mathrm{G}}(\xi_{\mathrm{th}}) \overset{\text{Gauss}}{=} 1 - (1 - \mathrm{e}^{-\xi_{\mathrm{th}}})^{D} \ , \tag{4}$$

which leads to

$$C_{\mathrm{s}}^{\mathrm{G}}(\xi_{\mathrm{th}}) \overset{\text{Gauss}}{=} (1 - (1 - \mathrm{e}^{-\xi_{\mathrm{th}}})^{D})^{U} \ . \tag{5}$$

Noteworthy, for Gaussian symbols, infinitely high signal peaks and hence an infinite PAR ($\xi_{\max} \to \infty$) is possible.

[2] In this paper, we quantify the computational complexity by considering the number of assessed signal candidates. The overall complexity of SLM is then the number of assessed candidates times the complexity for generating a signal candidate and calculating the respective PAR values. In [24], [19], [11], [15], variants of SLM have been proposed, which reduce the complexity of generating and evaluating signal candidates. Since we study methods for reducing the number of assessed candidates, the individual complexities are not of interest here. Moreover, the successive approach of can be combined with the other variants of SLM, too.

Using the relation $1-(1-\mathrm{e}^{-x})^y < y\mathrm{e}^{-x}$, which holds[3] for $y \geq 2$ and $10\log_{10}(x) \leq 12.7$ dB, the upper bound

$$C_{\mathrm{s}}^{\mathrm{G}}(\xi_{\mathrm{th}}) < C_{\mathrm{s,u}}^{\mathrm{G}}(\xi_{\mathrm{th}}) \overset{\mathrm{def}}{=} D^U \mathrm{e}^{-\xi_{\mathrm{th}} U} \tag{6}$$

results. In logarithmic scale, this expression corresponds to the linear function

$$\log(C_{\mathrm{s,u}}^{\mathrm{G}}(\xi_{\mathrm{th}})) = U \cdot (\log(D) - \xi_{\mathrm{th}}) \; . \tag{7}$$

Independent of U, $\log(C_{\mathrm{s,u}}^{\mathrm{G}}(\xi_{\mathrm{th}}))$ becomes zero ($C_{\mathrm{s,u}}^{\mathrm{G}}(\xi_{\mathrm{th}}) \overset{!}{=} 1$) at

$$\xi_{\mathrm{th}} = \log(D) \overset{\mathrm{def}}{=} \xi_{\infty}^{\mathrm{G}} \; . \tag{8}$$

As only probabilities less than one make sense, the bound (7) is only meaningful for $\xi_{\mathrm{th}} \geq \xi_{\infty}^{\mathrm{G}}$ and can further be extended to

$$\log(C_{\mathrm{s,u}}^{\mathrm{G}}(\xi_{\mathrm{th}})) = \begin{cases} 0 & , \; \xi_{\mathrm{th}} < \xi_{\infty}^{\mathrm{G}} \\ U(\xi_{\infty}^{\mathrm{G}} - \xi_{\mathrm{th}}) & , \; \xi_{\mathrm{th}} \geq \xi_{\infty}^{\mathrm{G}} \end{cases} \; . \tag{9}$$

The slope of this upper bound is given by U, the number of assessed candidates. Hence, the more candidates are taken into account, the steeper gets the slope.

In turn, spending infinite complexity, i.e., $U \to \infty$, the ccdf for Gaussian signaling $C_{\mathrm{s}}^{\mathrm{G}}(\xi_{\mathrm{th}})$ is upper bounded by the stair-step function

$$C_{\mathrm{s,u}}^{\mathrm{G}}(\xi_{\mathrm{th}}) = \begin{cases} 1 & , \; \xi_{\mathrm{th}} < \xi_{\infty}^{\mathrm{G}} \\ 0 & , \; \xi_{\mathrm{th}} \geq \xi_{\infty}^{\mathrm{G}} \end{cases} \quad \text{for } U \to \infty \; . \tag{10}$$

In words, investing an infinite complexity, it is guaranteed to find a candidate OFDM frame with PAR not exceeding $\xi_{\infty}^{\mathrm{G}} = \log(D)$; larger PAR values can asymptotically completely be eliminated. Using other approaches, this fact has already been observed in [29], [38], [25], where the PAR value $\xi_{\infty}^{\mathrm{G}}$ is referred to as *critical PAR*.

Interestingly, the mean PAR of the original signal is related to this critical PAR value. In order to calculate the mean value, we need the probability density function of the PAR, which is given by

$$f_{\xi}(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(1 - C_{\mathrm{o}}^{\mathrm{G}}(\xi)\right) = D\mathrm{e}^{-\xi}(1-\mathrm{e}^{-\xi})^{D-1}. \tag{11}$$

The mean PAR now calculates to ($\Psi(x)$ is the Digamma function and $\gamma = 0.5772\ldots$ is Euler's constant)

$$\begin{aligned} \mathrm{E}\{\xi\} &= \int_0^{\infty} \xi \cdot D\mathrm{e}^{-\xi}(1-\mathrm{e}^{-\xi})^{D-1} \, \mathrm{d}\xi \overset{z=\mathrm{e}^{-\xi}}{=} -D \cdot \int_0^1 \log(z)(1-z)^{D-1} \, \mathrm{d}z \\ &\overset{[16]}{=} \Psi(D+1) + \gamma \overset{[36]}{=} \sum_{k=1}^{D} \frac{1}{k} \approx \log(D) + \gamma = \xi_{\infty}^{\mathrm{G}} + \gamma. \end{aligned} \tag{12}$$

[3]For larger values of x the difference between left and right hand side expression alternates and tends to zero as $x \to \infty$.

The value ξ_∞^{G} of tolerable PAR values is derived from the upper bound (9). However, using the exact expression of the ccdf (5) and assuming that an infinite number U of signal candidates is assessable ($U \to \infty$), the ccdf equals zero for any value $\xi_{\mathrm{th}} \geq 1$. Thus, it is not only possible to find an alternative signal representation exhibiting a PAR less than any threshold $\xi_{\mathrm{th}} \geq \xi_\infty^{\mathrm{G}}$ but also for any threshold $1 \leq \xi_{\mathrm{th}} < \xi_\infty^{\mathrm{G}}$. The benefit of the bound ξ_∞^{G} can be seen as follows. For values $\xi_{\mathrm{th}} \geq \xi_\infty^{\mathrm{G}}$ the ccdf in logarithmic scale decreases (at least) linearly to $-\infty$ with increasing ξ_{th}. Now, increasing the number U of assessed signal candidates even accelerates the convergence to $-\infty$. Contrary, for values $\xi_{\mathrm{th}} < \xi_\infty^{\mathrm{G}}$ looking at (5) in logarithmic scale leads to $\log(C_{\mathrm{s}}^{\mathrm{G}}(\xi_{\mathrm{th}})) = U \cdot \log(1 - (1 - \mathrm{e}^{-\xi_{\mathrm{th}}})^D) = U \cdot \log(1 - \varepsilon)$ with positive but very small ε. It is obvious, that the convergence of the ccdf in logarithmic scale to $-\infty$ is very slow in this region.

C. QAM Signalling with Oversampling

In practical communication systems, the elements of the OFDM symbol $\boldsymbol{A} = [A_d]$ are usually drawn from an M-ary QAM constellation, whereby all elements can be regarded to be pairwise statistically independent. The transformation into time domain via the IDFT describes a D-fold superposition of all these elements. Due to this reason and according to the central limit theorem, the distribution of time-domain signal matches very well a Gaussian distribution. However, the PAR of the time-domain signal at Nyquist rate is not the relevant metric to represent the contribution of the actual OFDM symbol to the out-of-band radiation. For this reason the PAR of the continuous-time (after pulse shaping) signal ought to be assessed. In order to reflect the PAR of the continuous-time signal oversampling is applied [33], [34]. According to [37] an oversampling factor $L = 2$ is sufficient to get a well approximate of the continuous-time PAR. Nevertheless, the following analysis uses oversampling factors of $L = 2, 4, 8$.

The derivations from the previous subsection are equally valid for any kind of OFDM signaling. As long as $\log(C_{\mathrm{o}}(\xi_{\mathrm{th}}))$ (the ccdf of the original signal; not necessarily for Gaussian samples at Nyquist rate) is upper bounded by a straight line with zero crossing at a specific point and some slope, the value ξ_∞ can be calculated. In [37], [25] an upper bound for the ccdf of the continuous-time signal has been derived, which shows this behaviour. Thus, the typical behaviour of the ccdf of PAR incorporating oversampling can be approximated by an upper bound as given in (9). As all signal candidates are assumed to be statistically independent, the slope of this upper bound is still proportional to the number U of alternative signal representations. However, the characteristic point of this upper bound (the point ξ_∞, where it is non-differentiable) and possibly a factor in the slope change with different oversampling factors.

In order to calculate the values of ξ_∞ for the respective oversampling factor, numerical simulations of the peak-power distribution were conducted. The natural logarithm ($\log$) of the resulting ccdf was approximated by a straight line (in the sense of least squared deviation) in the region $10^{-3} < \log(\text{ccdf}(\xi_{\text{th}})) < 10^{-4}$.

Fig. 1 shows $\log(\text{ccdf}(\xi_{\text{th}}))$ for different oversampling factors ($L = 1, 2, 4, 8$). The upper bound of the continuous-time signal is taken from [38, Eq. (14)]. Please notice, unlike the conventional ccdf–ξ_{th}–diagram, the abscissa is plotted in linear scale.

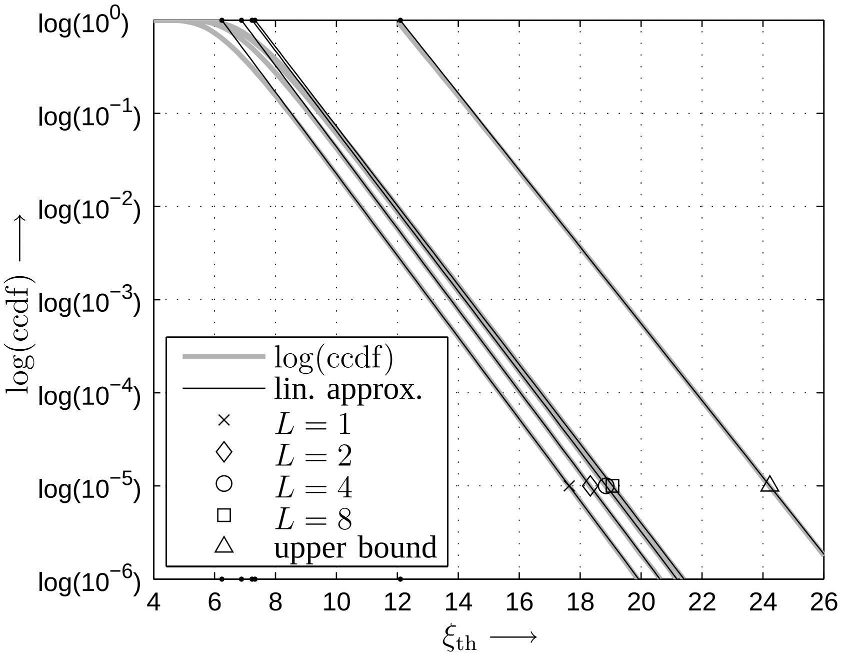


Fig. 1. Natural logarithm of ccdf of ξ_{th} (in linear scale) of the original signal (i.e., $U = 1$) for different oversampling factors $L = 1, 2, 4, 8$ (gray). Additionally, the upper bound of the ccdf for continuous-time signals [38] is plotted. The linear approximations (in the least-squared sense) of these curves are depicted black. The region, where this approximation is accomplished is marked gray. The intersection points of the approximation at $\log(\text{ccdf}) = 0$ are marked by a black dot (which is repeated at the abscissa). $D = 512$, ($M = 4$)-QAM signaling per carrier.

The resulting characteristic points ξ_∞ for various oversampling factors are shown in Table I (the resulting slope of the curves is always very close to -1; values ranging from -0.98 to -1.01 were observed). The top table compares ξ_∞ for different sizes M of the modulation alphabet at a fixed number of subcarriers $D = 512$. As it can be seen from this table, the limit ξ_∞ is independent from the modulation alphabet M. Increasing the oversampling factor L leads to higher values of ξ_∞, whereby a loss of approximately 0.7dB can be observed. The bottom table compares the resulting characteristic points ξ_∞ for different numbers D of subcarriers (here $M = 4$). As a larger number of subcarriers leads to higher PAR values, the resulting characteristic points increase as well. The respective loss from $L = 1$ to $L = 8$ is about $0.7\,\text{dB}$ to $0.9\,\text{dB}$ but decreases with an increasing number of subcarriers.

TABLE I. VALUES OF ξ_∞ (IN dB) DEPENDING ON OVERSAMPLING FACTOR L AND TOP: THE MODULATION ALPHABET M ($D = 512$), BOTTOM: THE NUMBER OF SUBCARRIERS D ($M = 4$).

$D = 512$

M \ L	1	2	4	8	Gaussian ($L=1$)	upper bound [38]
4	7.95	8.38	8.60	8.65	7.95	10.83
16	7.96	8.39	8.61	8.66		10.83
64	7.95	8.39	8.61	8.66		10.83
256	7.95	8.39	8.60	8.66		10.83

$M = 4$

D \ L	1	2	4	8	Gaussian ($L=1$)	upper bound [38]
64	6.14	6.73	6.96	7.02	6.19	9.94
128	6.79	7.32	7.56	7.61	6.86	10.26
256	7.42	7.89	8.11	8.18	7.44	10.56
512	7.95	8.38	8.60	8.65	7.95	10.83

IV. SUCCESSIVE SELECTED MAPPING

A. *Successive Selected Mapping with Infinite Number of Candidates*

Based on the insight that a limit (10) for the upper bound on the PAR exists, a new version of selected mapping can be designed. OFDM frames with PAR below the threshold ξ_∞ should be readily accepted for transmission. Given frames with $\xi > \xi_\infty$, alternative representations are tested until the PAR falls below the threshold ξ_∞. Now, the number U of assessed candidates is different for each OFDM frame. As this is done successively, we denote this strategy as *successive SLM (SSLM)*. In the first approach U is not limited and may tend to infinity.

The results derived subsequently are not only valid if the threshold ξ_∞ is chosen. All results remain valid if any value $\xi_0 > 1$ is the desired PAR limit. In all equations ξ_∞ has to be replaced by ξ_0 and C_∞ by $C_0 \stackrel{\text{def}}{=} C_\text{o}(\xi_0)$, respectively.

The probability that a given OFDM frame exceeds the threshold ξ_∞ reads

$$\Pr\{\xi > \xi_\infty\} = C_\text{o}(\xi_\infty) \stackrel{\text{def}}{=} C_\infty \; , \tag{13}$$

and for Gaussian samples we have

$$\Pr\{\xi > \xi_\infty^{\mathrm{G}}\} = C_{\mathrm{o}}^{\mathrm{G}}(\xi_\infty^{\mathrm{G}}) \stackrel{\mathrm{def}}{=} C_\infty^{\mathrm{G}} = 1 - \left(\frac{D-1}{D}\right)^D \ . \tag{14}$$

Hence, the probability that u different signal representations have to be assessed to find an OFDM frame with PAR less than ξ_∞ is given by the Geometric distribution [35]

$$\Pr\{U = u\} = C_\infty^{u-1} \cdot (1 - C_\infty) \ . \tag{15}$$

On average, the number of signal representations, which have to be tested in order to achieve a PAR less than ξ_∞, is given by[4]

$$\begin{aligned} \mathrm{E}\{U\} &= \sum_{u=1}^{\infty} u \cdot \Pr\{U = u\} \quad \left[\frac{\text{assessed cand.}}{\text{OFDM frame}}\right] \\ &= \frac{1}{1 - C_\infty} \qquad (16) \\ &\stackrel{\text{Gauss}}{=} \left(\frac{D}{D-1}\right)^D \ . \qquad (17) \end{aligned}$$

Interestingly, as can be seen from (16), $\mathrm{E}\{U\}$ is simply the reciprocal of the cdf of PAR. Moreover, for Gaussian signaling this expected value converges to Euler's number e if the length D of the OFDM frames grows to infinity

$$\lim_{D\to\infty} \left(\frac{D}{D-1}\right)^D = \mathrm{e} = 2.71\ldots \ , \tag{18}$$

and is only slightly larger for practical values of D.

The variance of the assessed number of signal candidates can be calculated to

$$\mathrm{Var}\{U\} = \frac{C_\infty}{(1 - C_\infty)^2} \stackrel{\text{Gauss}}{=} \left(\frac{D}{D-1}\right)^D \left(\left(\frac{D}{D-1}\right)^D - 1\right) \stackrel{D\to\infty}{\longrightarrow} \mathrm{e}(\mathrm{e}-1). \tag{19}$$

The ccdf of PAR when performing SSLM is given by

$$\begin{aligned} \mathrm{ccdf}_{\mathrm{SSLM}}(\xi_{\mathrm{th}}) &= \Pr\{\xi > \xi_{\mathrm{th}} \,|\, \xi \le \xi_\infty\} = \frac{\Pr\{\xi_{\mathrm{th}} < \xi \le \xi_\infty\}}{\Pr\{\xi < \xi_\infty\}} \\ &= \begin{cases} \frac{C_{\mathrm{o}}(\xi_{\mathrm{th}}) - C_\infty}{1 - C_\infty} & \xi_{\mathrm{th}} < \xi_\infty \\ 0 & \xi_{\mathrm{th}} \ge \xi_\infty \end{cases} \ . \qquad (20) \end{aligned}$$

In order to decode the information correctly, the receiver must be aware of the applied mapping. In SSLM the additional information is equivalent to the number of trials to achieve an OFDM frame with PAR lower than ξ_∞.

[4]A similar analysis has already been carried out in [4]. However, the compact result (16) and the interpretation with the Gaussian approximation (17) have not been observed.

As the number of required trials (15) is not uniformly distributed, entropy coding (e.g., Huffman coding [9]) can be applied. Assuming an optimal source coding, the average number of side information bits per OFDM frame necessary to encode the number of assessed candidates is given by the entropy

$$\begin{aligned}
H_\infty(U) &= -\sum_{u=1}^{\infty} \Pr\{U=u\} \cdot \log_2\left(\Pr\{U=u\}\right) && (21)\\
&= -\sum_{u=1}^{\infty} C_\infty^{u-1}(1-C_\infty) \cdot \log_2\left(C_\infty^{u-1}(1-C_\infty)\right)\\
&= -(1-C_\infty)\sum_{u=0}^{\infty}\left(\log_2(C_\infty)\cdot uC_\infty^u + \log_2(1-C_\infty)\cdot C_\infty^u\right)\\
&= -(1-C_\infty)\left(\log_2(C_\infty)\cdot\frac{C_\infty}{(1-C_\infty)^2} + \frac{\log_2(1-C_\infty)}{1-C_\infty}\right)\\
&= \frac{\mathrm{e}_2(C_\infty)}{1-C_\infty}\left[\frac{\text{bit}}{\text{OFDM frame}}\right]\,. && (22)
\end{aligned}$$

Assuming again Gaussian signaling, if the number of subcarriers D grows to infinity the entropy tends to $\mathrm{e}\cdot\mathrm{e}_2(1/\mathrm{e}) = 2.57\ldots < \mathrm{e}$. For practical values D, as long as $C_\infty < 0.7$, the average number of side-information bits is always less than 3 bits per OFDM frame.

The variance of the entropy can be calculated to

$$\begin{aligned}
&\mathrm{Var}\{-\log_2(\Pr\{U=u\})\} =\\
&= \sum_{u=1}^{\infty} C_\infty^{u-1}(1-C_\infty)\left(\log_2(C_\infty^{u-1}(1-C_\infty))\right)^2 - \left(\frac{\mathrm{e}_2(C_\infty)}{1-C_\infty}\right)^2\\
&= (1-C_\infty)(\log_2(C_\infty))^2\cdot\sum_{u=0}^{\infty}u^2C_\infty^u + (1-C_\infty)\log_2(C_\infty)\log_2(1-C_\infty)\cdot\sum_{u=0}^{\infty}uC_\infty^u\\
&\quad +(1-C_\infty)(\log_2(1-C_\infty))^2\cdot\sum_{u=0}^{\infty}C_\infty^u - \left(\frac{\mathrm{e}_2(C_\infty)}{1-C_\infty}\right)^2\\
&= (1-C_\infty)(\log_2(C_\infty))^2\cdot\frac{C_\infty(1+C_\infty)}{(1-C_\infty)^3} + (1-C_\infty)\log_2(C_\infty)\log_2(1-C_\infty)\cdot\frac{C_\infty}{(1-C_\infty)^2}\\
&\quad +(1-C_\infty)(\log_2(1-C_\infty))^2\cdot\frac{1}{1-C_\infty} - \left(\frac{\mathrm{e}_2(C_\infty)}{1-C_\infty}\right)^2\\
&= C_\infty\cdot\left(\frac{\log_2(C_\infty)}{1-C_\infty}\right)^2 && (23)
\end{aligned}$$

B. *Successive Selected Mapping with Limited Number of Candidates*

In a practical system it is not possible to implement SSLM as described in the previous section because the number of candidates which have to be assessed for one OFDM frame might be infinitely large. A straightforward approach for implementing successive SLM is to limit the maximum number of assessed signal candidates to $\hat{U}$. The first candidate which exhibits a PAR less than ξ_∞ is chosen for transmission and no other candidate is considered. If none of the $\hat{U}$ candidates fulfils this condition that one exhibiting the lowest PAR is transmitted. Related approaches, which also define a PAR threshold, have been proposed in [30], [5].

The ccdf of this strategy is given as follows: For $\xi_{\mathrm{th}} < \xi_\infty$ the ccdf consists of two parts. On the one hand, if at least one out of the at most $\hat{U}$ considered OFDM frames is less than the threshold ξ_∞ (probability $1 - C_\infty^{\hat{U}}$) the ccdf is given by (20). On the other hand, if all $\hat{U}$ assessed OFDM frames are greater than ξ_∞ (probability $C_\infty^{\hat{U}}$) the ccdf corresponds to 1. For $\xi_{\mathrm{th}} \geq \xi_\infty$, i.e., the best PAR after evaluating the maximum possible number $\hat{U}$ of candidates is still greater than the chosen limit ξ_∞, the ccdf corresponds to the one of SLM, i.e., (3) with $U = \hat{U}$. Hence, the ccdf of SSLM reads

$$\mathrm{ccdf}_{\mathrm{SSLM}}(\xi_{\mathrm{th}}) = \begin{cases} \frac{C_{\mathrm{o}}(\xi_{\mathrm{th}}) - C_\infty}{1 - C_\infty}(1 - C_\infty^{\hat{U}}) + C_\infty^{\hat{U}} & , \xi_{\mathrm{th}} < \xi_\infty \\ (C_{\mathrm{o}}(\xi_{\mathrm{th}}))^{\hat{U}} & , \xi_{\mathrm{th}} \geq \xi_\infty \end{cases} . \tag{24}$$

In order to calculate the average number of assessed candidates and the required number of side information bits, we have to distinguish between two cases. The probability $\mathrm{Pr_c}\{U = u\}$ that u candidates are assessed for a given OFDM frame is again determined by a Geometric distribution (15). For $u = \hat{U}$ the probability is given by the sum of (15) over $u = \hat{U}, \ldots, \infty$:

$$\mathrm{Pr_c}\{U = u\} = \begin{cases} C_\infty^{u-1}(1 - C_\infty) & , u < \hat{U} \\ \sum_{\tilde{u}=\hat{U}}^{\infty} C_\infty^{\tilde{u}-1}(1 - C_\infty) = C_\infty^{\hat{U}-1} & , u = \hat{U} \end{cases} , \tag{25}$$

Hence, the expected value of evaluated candidates is given by

$$\mathrm{E}\{U\} = \frac{1 - C_\infty^{\hat{U}}}{1 - C_\infty} \left[\frac{\text{assessed cand.}}{\text{OFDM frame}}\right] < \frac{1}{1 - C_\infty} , \tag{26}$$

and the variance reads

$$\mathrm{Var}\{U\} = \frac{C_\infty}{(1 - C_\infty)^2} - C_\infty^{\hat{U}} \cdot \frac{C_\infty^{\hat{U}} + (1 - C_\infty)(2\hat{U} - 1)}{(1 - C_\infty)^2} < \frac{C_\infty}{(1 - C_\infty)^2}. \tag{27}$$

Both results are upper bounded by the ones for $\hat{U} \to \infty$ ((16) or (19)).

The probability $\mathrm{Pr_i}\{U = u\}$ that the u^{th} OFDM frame is chosen for transmission and thus the u^{th} index has to be transmitted as side information is not equal to (25). If the PAR of the $\hat{U}^{\mathrm{th}}$ assessed candidate is still above the threshold ξ_∞, the best one among all $\hat{U}$ previously assessed OFDM frames, which is uniformly distributed, is selected for transmission. Hence, the probability reads

$$\begin{aligned} \mathrm{Pr_i}\{U = u\} &= C_\infty^{u-1}(1 - C_\infty) + \frac{1}{\hat{U}} \sum_{\tilde{u}=\hat{U}+1}^{\infty} C_\infty^{\tilde{u}-1}(1 - C_\infty) \\ &= C_\infty^{u-1}(1 - C_\infty) + \frac{C_\infty^{\hat{U}}}{\hat{U}} , \quad u = 1, \ldots, \hat{U} , \end{aligned} \tag{28}$$

and the entropy of U amounts to (an analytical simplification is not possible here)

$$H_{\hat{U}}(U) = -\sum_{u=1}^{\hat{U}} \left(C_\infty^{u-1}(1-C_\infty) + \frac{C_\infty^{\hat{U}}}{\hat{U}} \right) \log_2 \left(C_\infty^{u-1}(1-C_\infty) + \frac{C_\infty^{\hat{U}}}{\hat{U}} \right) \left[\frac{\text{bit}}{\text{OFDM frame}} \right] \quad (29)$$

$$< \frac{\mathrm{e}_2(C_\infty)}{1-C_\infty} \quad (30)$$

Comparing both results, $\mathrm{E}\{U\}$ and $H_{\hat{U}}(U)$, with the ones for $\hat{U} \to \infty$, the average number of assessed candidates and the minimum number of required side-information bits per OFDM frame is strictly lower than for $\hat{U} \to \infty$ given in (16) or (22) (a proof of (30) is given in the Appendix).

C. Numerical Results

The top row of Fig. 2 shows the ccdf of SSLM for $\hat{U} = 1$ (original signal) and $\hat{U} = 2, 4, 8, 16$ (left: no oversampling, i.e., $L = 1$; right: $L = 8$). The threshold for tolerable PAR values is chosen to $\xi_0 = \xi_\infty$ and the theoretic ccdf curves are determined after (20) and (24). In the left plot, the theoretic results are based on the Gaussian assumption, i.e., $\xi_0 = \xi_\infty^{\mathrm{G}} = \log(D)$ and the ccdf of the original signal is given according to (4). In the right plot of Fig. 2 ($L = 8$) the theoretic results of the ccdf of SSLM are based on numerical simulations of the PAR distribution of the original signal. In both cases, theoretical results and numerical simulations fit very well.

The two bottom rows of Fig. 2 show the ccdf results of SSLM for different values of ξ_0. In the middle row, the stopping threshold is chosen to $10\log_{10}(\xi_0) = 10\log_{10}(\xi_\infty) - 0.25\,\mathrm{dB}$, in the bottom row to $10\log_{10}(\xi_0) = 10\log_{10}(\xi_\infty) + 0.25\,\mathrm{dB}$. Choosing the threshold ξ_0 lower than ξ_∞ results almost in the same ccdf as of pure SLM according to (3). In each case, for values $\xi_{\mathrm{th}} \geq \xi_0$, the ccdf of SSLM is equivalent to the one of pure SLM given in (3). If the threshold ξ_0 is chosen to values greater than ξ_∞ a strong concentration of PAR values less than the threshold occurs.

Fig. 3 shows the probabilities according to (15) of the number u of OFDM frames to be assessed to find a candidate below $\xi_0 = \xi_\infty$. These probabilities are depicted for Gaussian signaling, 4-QAM modulation at Nyquist-rate sampling and with eight times oversampling. Even if there are some slight deviations between these different cases, all probabilities decrease pretty fast to zero with increasing u, which explains the low average value of U.

In Fig. 4 numerical results for the average number of assessed signal candidates (top row) and required bits of side information (bottom row) depending on the stopping threshold ξ_0 are depicted. Theoretic results for $\mathrm{E}\{U\}$ are given according to (16) and (26) and for $H(U)$ according to (22) and (29). The left column shows results for Nyquist-rate sampling, whereas the right column an oversampling factor of $L = 8$ is applied (respective theory curves base on numerical results of original signal). From this plot the relevance of the threshold $\xi_0 = \xi_\infty$ becomes clear. For values $\xi_0 \geq \xi_\infty$ the average number of assessed candidates $\mathrm{E}\{U\}$ and the number of required side information bits $H(U)$, respectively, are almost independent on the maximum number $\hat{U}$ of assessed signal candidates and are pretty close to the theoretic curves for $\hat{U} \to \infty$. In addition to that, they converge fast to its optimal values of $\min_{\xi_0} \mathrm{E}\{U\} = 1$ and $\min_{\xi_0} H(U) = 0$. For $\xi_0 < \xi_\infty$ the curves increase significantly and tend quickly to their maximum value $\max_{\xi_0} \mathrm{E}\{U\} = \hat{U}$ and $\max_{\xi_0} H(U) = \log_2(\hat{U})$.

D. Design Strategy for Successive Selected Mapping

So far, the design parameters of SSLM, $\hat{U}$ and tolerated PAR ξ_0, were arbitrarily given. A design rule closer to practical demands would be to predetermine the threshold ξ_0 and a probability Pr_0 with which this threshold might be exceeded at most. The point (ξ_0, Pr_0) specifies a mask in the ξ_{th}–ccdf–diagram, which defines an allowed region for the desired ccdf curve.

For values $\xi_{\mathrm{th}} \geq \xi_0$ the ccdf of SSLM is determined by the behaviour of SLM assessing $\hat{U}$ candidates. Now, given the parameters Pr_0 and ξ_0, the parameter $\hat{U}$ has to be chosen that for $\xi > \xi_0$ the ccdf of SLM (5) is always less than Pr_0; in the worst case we have

$$\mathrm{Pr}\{\xi > \xi_0\} = (C_\mathrm{o}(\xi_0))^{\hat{U}} \leq \mathrm{Pr}_0 \ . \tag{31}$$

Hence, the maximum number of required signal candidates is given by

$$\hat{U} = \left\lceil \frac{\log(\mathrm{Pr}_0)}{\log(C_\mathrm{o}(\xi_0))} \right\rceil . \tag{32}$$

Fig. 5 shows the resulting values of $\mathrm{E}\{U\}$ (top row) and $H(U)$ (bottom row) in a contour diagram over ξ_0 and Pr_0. For convenience, we restricted ourselves to QAM signalling at Nyquist rate. However, the similar results hold for the oversampled signal. The regions for different values of $\hat{U}$ are delimited by dashed lines. From here we can see, that Pr_0 influences $\mathrm{E}\{U\}$ and $H(U)$ only marginally. The average number of assessed candidates and number of redundancy bits is mostly governed by the tolerated peak-power threshold ξ_0.

E. Comparison with Absolute Bound on Redundancy

In [28], [27] a theoretic PAR reduction technique has been addressed, which is optimum in the sense of required redundancy. This scheme has been derived for Gaussian signalling at Nyquist rate, however using the above considerations it is equally applicable for QAM signalling.

Imagine a list of all possible OFDM frames sorted with respect to their PAR. Now, we choose a threshold ξ_0 such that the number of OFDM frames above and below this threshold is equal. Transmitting only frames below the threshold, i.e., the half, one bit of redundancy is spent. In general,

R bits of redundancy are present if a fraction of $1-2^{-R} \stackrel{!}{=} C_{\mathrm{o}}(\xi_0)$ OFDM frames is not used. Hence, the number of redundancy bits is given by

$$R = -\log_2(1 - C_{\mathrm{o}}(\xi_0)) \left[\frac{\text{bit}}{\text{OFDM frame}}\right] . \tag{33}$$

Noteworthy, the complexity of this scheme is far from being practical as look-up tables with mappings of all possible OFDM frames must be known to both, transmitter and receiver.

Fig. 6 compares this bound with the required redundancy in terms of the entropy for SSLM with $\hat{U} \to \infty$. The practical implementation of SSLM according to Sec. IV-D and pure SLM are also depicted in Fig. 6 at a tolerated clipping probability $\mathrm{Pr}_0 = 10^{-4}$ at $\xi_{\mathrm{th}} = \xi_0$. The curve of SSLM lies very close to the bound (33) as it requires less redundancy than SSLM with $\hat{U} \to \infty$. Evidently, this scheme outperforms pure SLM. Hence it can be stated that SSLM utilizes the implicit redundancy in an almost optimum way, but at the same time is computationally efficient.

V. Conclusions

Selected mapping for peak-to-average power ratio reduction in OFDM transmission has been investigated. In particular, the asymptotic behaviour for an infinite number U of assessed signal candidates has been analyzed. Based on the assumption of Gaussian distributed time-domain signals and on a linear approximation of the ccdf, a (practical) upper bound for maximum tolerable PAR values of $\xi_\infty^{\mathrm{G}} = \log(D)$ has been derived. Although, an infinite number U of assessed signal candidates would lead (theoretically for Gaussian signalling) to a PAR of $\xi = 1$ this bound has practical relevance.

For the analysis of practical systems it is not sufficient to analyse the signal at Nyquist rate. In order to obtain an accurate estimate of the PAR of the actual continuous-time transmit signal an oversampled signal has to be analyzed [33], [34]. Unfortunately, there does not exist an analytical expression of the ccdf for such signals. We have shown in this paper, that all given analytical results are also valid for any kind of oversampling, once the ccdf of the respective original OFDM signal without PAR reduction is known.

Furthermore, a novel PAR reduction scheme, named SSLM, has been derived, which does not try to reduce the PAR as low as possible but only tries to get below a given threshold ξ_0. The analysis in this paper showed that this threshold should be chosen in the region slightly larger than ξ_∞.

The huge benefit of SSLM is the significant decrease in required complexity and redundancy. As a rule of thumb, an average number of assessed candidates (much) less than $\mathrm{e} = 2.71\ldots$ and required bits of side information (much) less than $\mathrm{e}\cdot\mathrm{e}_2(1/\mathrm{e}) = 2.57\ldots$ are achievable. In practical systems, the savings in computational complexity lead to a reduction in required battery power, which is desirable for mobile systems, for instance. In order to further reduce the computational complexity, SSLM may straightforwardly be combined with low complex variants of SLM, proposed, e.g., in [24], [19], [11], [15].

All PAR reduction schemes have in common, that they influence the transmit signal by introducing computational complexity and, in addition to that, either redundancy and/or an increased transmit power. Many PAR reduction schemes suffer from these issues. For instance, with tone reservation [33] a convex optimization problem [6] has to be solved. Although reduced-complexity solutions are known [33], [22], [1], the computational effort is quite high. In addition to that the redundancy, i.e., subcarriers which are not used for transmitting information but for PAR reduction, is quite high, too. In contrast to these issues the novel technique SSLM shows that almost any desired PAR reduction performance is achievable by introducing only little complexity and redundancy and by not influencing average transmit power.

APPENDIX

In order to prove the relation (30), we show that the difference between the larger and the smaller term is greater than zero.

$$\begin{aligned}
&\frac{\mathrm{e}_2(C_\infty)}{1-C_\infty} - H_{\hat{U}}(U) = \\
= \quad & \frac{\mathrm{e}_2(C_\infty)}{1-C_\infty} + \sum_{u=1}^{\hat{U}} \left(C_\infty^{u-1}(1-C_\infty) + \frac{C_\infty^{\hat{U}}}{\hat{U}} \right) \log_2 \left(C_\infty^{u-1}(1-C_\infty) + \frac{C_\infty^{\hat{U}}}{\hat{U}} \right) \\
> \quad & \frac{\mathrm{e}_2(C_\infty)}{1-C_\infty} + \sum_{u=1}^{\hat{U}} \left(C_\infty^{u-1}(1-C_\infty) + \frac{C_\infty^{\hat{U}}}{\hat{U}} \right) \log_2 \left(C_\infty^{u-1}(1-C_\infty) \right) \\
= \quad & \frac{\mathrm{e}_2(C_\infty)}{1-C_\infty} + \sum_{u=1}^{\hat{U}} \left(C_\infty^{u-1}(1-C_\infty) + \frac{C_\infty^{\hat{U}}}{\hat{U}} \right) ((u-1)\log_2(C_\infty) + \log_2(1-C_\infty)) \\
= \quad & \frac{\mathrm{e}_2(C_\infty)}{1-C_\infty} + (1-C_\infty)\log_2(C_\infty) \sum_{u=0}^{\hat{U}-1} u C_\infty^u + \frac{C_\infty^{\hat{U}}}{\hat{U}} \log_2(C_\infty) \sum_{u=0}^{\hat{U}-1} u \\
& \qquad + (1-C_\infty)\log_2(1-C_\infty) \sum_{u=0}^{\hat{U}-1} C_\infty^u + C_\infty^{\hat{U}} \log_2(1-C_\infty) \\
= \quad & \frac{\mathrm{e}_2(C_\infty)}{1-C_\infty} + \log_2(C_\infty) \frac{C_\infty - C_\infty^{\hat{U}+1}}{1-C_\infty} - C_\infty^{\hat{U}} \log_2(C_\infty)\hat{U} + C_\infty^{\hat{U}} \log_2(C_\infty) \frac{\hat{U}-1}{2} \\
& \qquad + \log_2(1-C_\infty) - C_\infty^{\hat{U}} \log_2(1-C_\infty) + C_\infty^{\hat{U}} \log_2(1-C_\infty) \\
= \quad & \frac{\mathrm{e}_2(C_\infty)}{1-C_\infty} + \log_2(C_\infty) \frac{C_\infty - C_\infty^{\hat{U}+1}}{1-C_\infty} - C_\infty^{\hat{U}} \log_2(C_\infty) \frac{\hat{U}+1}{2} + \log_2(1-C_\infty) \\
= \quad & -C_\infty^{\hat{U}} \log_2(C_\infty) \left(\frac{C_\infty}{1-C_\infty} + \frac{\hat{U}+1}{2} \right)
\end{aligned}$$

Using the inequality $\log_2(a) \leq (a-1)\log_2(\mathrm{e})$ [14] the difference can further be manipulated

$$\begin{aligned}
\geq \quad & C_\infty^{\hat{U}}(1-C_\infty)\log_2(\mathrm{e}) \left(\frac{C_\infty}{1-C_\infty} + \frac{\hat{U}+1}{2} \right) \\
> \quad & C_\infty^{\hat{U}+1} \log_2(\mathrm{e}) > 0 \, .
\end{aligned} \tag{34}$$

Noteworthy, according to (34) the upper bound (30) can further be tightened to

$$H_{\hat{U}}(U) < \frac{\mathrm{e}_2(C_\infty)}{1-C_\infty} - C_\infty^{\hat{U}+1} \cdot 1.443 \, . \tag{35}$$

REFERENCES

[1] A. Aggarwal, E.R. Stauffer, and T.H. Meng. Optimal Peak-to-Average Power Ratio Reduction in MIMO-OFDM Systems. In *Proceedings of IEEE International Conference on Communications (ICC)*, Istanbul, Turkey, June 2006.

[2] J. Armstrong. Peak-to-Average Power Reduction for OFDM by Repeated Clipping and Frequency Domain Filtering. *IEE Electronics Letters*, 38(5):246–247, February 2002.

[3] R. Bäuml, R.F.H. Fischer, and J.B. Huber. Reducing the Peak-to-Average Power Ratio of Multicarrier Modulation by Selected Mapping. *IEE Electronics Letters*, 32(22):2056–2057, November 1996.

[4] R.J. Baxley and G.T. Zhou. Assessing Peak-to-Average Power Ratios for Communications Applications. In *Proceedings of IEEE MILCOM Conference*, October 2004.

[5] R.J. Baxley and G.T. Zhou. MAP Metric for Blind Phase Sequence Detection in Selected Mapping. *IEEE Transactions on Broadcasting*, 51(4):565–567, December 2005.

[6] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[7] M. Breiling, S. Müller-Weinfurtner, and J.B. Huber. SLM Peak-Power Reduction without Explicit Side Information. *IEEE Communications Letters*, 5(6):239–241, June 2001.

[8] N. Carson and T.A. Gulliver. Peak-to-average power ratio reduction of OFDM using repeat-accumulate codes and selective mapping. In *Proceedings IEEE International Symposium on Information Theory (ISIT)*, June 2002.

[9] T.M. Cover and J.A. Thomas. *Elements of Information Theory*. Wiley-Interscience, 2006.

[10] R.F.H. Fischer. Peak-to-Average Power Ratio (PAR) Reduction in OFDM Based on Lattice Decoding. In *Proceedings of the 11th Int. OFDM-Workshop*, Hamburg, August 2006.

[11] R.F.H. Fischer. Widely-Linear Selected Mapping for Peak-to-Average Power Ratio Reduction in OFDM. *IEE Electronics Letters*, 43(14):766–767, July 2007.

[12] R.F.H. Fischer and C. Siegl. OFDM Peak-to-Average Power Ratio Reduction based on the Simplex Code. In *Proceedings IEEE International Symposium on Information Theory (ISIT)*, Toronto, Canada, July 2008.

[13] R.F.H. Fischer and C. Siegl. Peak-to-Average Power Ratio Reduction in OFDM using Reed-Solomon Codes. In *International Zurich Seminar (IZS '08)*, Zurich, Switzerland, March 2008.

[14] R.G. Gallager. *Information Theory and Reliable Communications*. John Wiley & Sons, New York, 1968.

[15] A. Ghassemi and T.A. Gulliver. Partial Selective Mapping OFDM with Low Complexity IFFTs. *IEEE Communication Letters*, 12(1):4–6, January 2008.

[16] W. Gröbner and N. Hofreiter. *Integraltafel Zweiter Teil — Bestimmte Integrale*. Springer Verlag, Vienna, 1966. (In German).

[17] R. Habendorf, W. Rave, and G. Fettweis. Nonlinear Predistortion with Reduced Peak-to-Average Power Ratio. In *Proceedings of the International Symposium on Wireless Personal Multimedia Communications (WPMC'06)*, San Diego, USA, September 2006.

[18] S.H. Han and J.H. Lee. An Overview of Peak-to-Average Power Ratio Reduction Techniques for Multicarrier Transmission. *IEEE Wireless Communications*, 12(2):56–65, April 2005.

[19] S.-J. Heo, H.-S. Noh, J.-S. No, and D.-J. Shin. A Modified SLM Scheme With Low Complexity for PAPR Reduction of OFDM Systems. *IEEE Transactions on Broadcasting*, 53(4):804–808, December 2007.

[20] A.D.S. Jayalath and C. Tellambura. The Use of Interleaving to Reduce the Peak-to-Average Power Ratio of an OFDM Signal. In *Proceedings of IEEE Global Telecommunications Conference (GLOBECOM)*, pages 82–86, San Francisco, USA, November 2000.

[21] A.D.S. Jaylath and C. Tellambura. SLM and PTS Peak-Power Reduction of OFDM Signals Without Side Information. *IEEE Transactions on Wireless Communications*, 4(5):2006–2013, September 2005.

[22] B.S. Krongold and D.L. Jones. A New Tone Reservation Method for Complex-Baseband PAR Reduction in OFDM Systems. In *Proceedings of IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP)*, May 2002.

[23] B.S. Krongold and D.L. Jones. PAR Reduction in OFDM via Active Constellation Extension. *IEEE Transactions on Broadcasting*, 49(3):258–268, September 2003.

[24] D.-W. Lim, J.-S. No, C.-W. Lim, and H. Chung. A New SLM OFDM Scheme With Low Complexity for PAPR Reduction. *IEEE Signal Processing Letters*, 12(2):93–96, February 2005.

[25] S. Litsyn and G. Wunder. Generalized bounds on the crest-factor distribution of OFDM signals with applications to code design. *IEEE Transactions on Information Theory*, 52(3):992–1006, March 2006.

[26] S. Müller and J.B. Huber. OFDM with Reduced Peak-to-Average Power Ratio by Optimum Combination of Partial Transmit Sequences. *IEE Electronics Letters*, 33(5):368–369, February 1997.

[27] S.H. Müller, R.W. Bäuml, R.F.H. Fischer, and J.B. Huber. OFDM with Reduced Peak-to-Average Power Ratio by Multiple Signal Representation. *Annals of Telecommunications*, 52(1-2):58–67, February 1997.

[28] S. Müller-Weinfurtner. *OFDM for Wireless Communications*. PhD thesis, Universität Erlangen-Nürnberg, 2000.

[29] K.G. Paterson and V. Tarokh. On the Existence and Construction of Good Codes With Low Peak-to-Average Power Ratios. *IEEE Transactions on Information Theory*, 46(6):1974–1987, September 2000.

[30] H. Qian, C. Xiao, N. Chen, and G.T. Zhou. Dynamic Selected Mapping for OFDM. In *Proceedings of IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP)*, March 2005.

[31] K.-U. Schmidt. On the PMEPR of Phase-Shifted Binary Codes. In *Proceedings IEEE International Symposium on Information Theory (ISIT)*, Seattle, July 2006.

[32] K.-U. Schmidt. *On spectrally bounded codes for multicarrier communications*. PhD thesis, Technische Universität Dresden, 2007.

[33] J. Tellado. *Peak to Average Power Reduction for Multicarrier Modulation*. PhD thesis, Stanford University, 2000.

[34] C. Tellambura. Computation of the Continuous-Time PAR of an OFDM Signal with BPSK Subcarriers. *IEEE Communications Letters*, 5(5):185–187, May 2001.

[35] Y. Viniotis. *Probability and Random Processes for Electrical Engineers*. McGraw-Hill, 1998.

[36] E.W. Weisstein. *CRC Concise Encyclopedia of Mathematics*. Chapman & Hall/CRC, 2003.

[37] G. Wunder and H. Boche. Peak Value Estimation of Bandlimited Signals from their Samples, Noise Enhancement, and a Local Characterization in the Neighborhood of an Extremum. *IEEE Transactions on Signal Processing*, 51(3):771–780, March 2003.

[38] G. Wunder and H. Boche. Upper bounds on the statistical distribution of the crest-factor in OFDM transmission. *IEEE Transactions on Information Theory*, 49(2):488–494, February 2003.

[39] G.T. Zhou and L. Peng. Optimality Condition for Selected Mapping in OFDM. *IEEE Transactions on Signal Processing*, 54(8):3159–3165, August 2006.

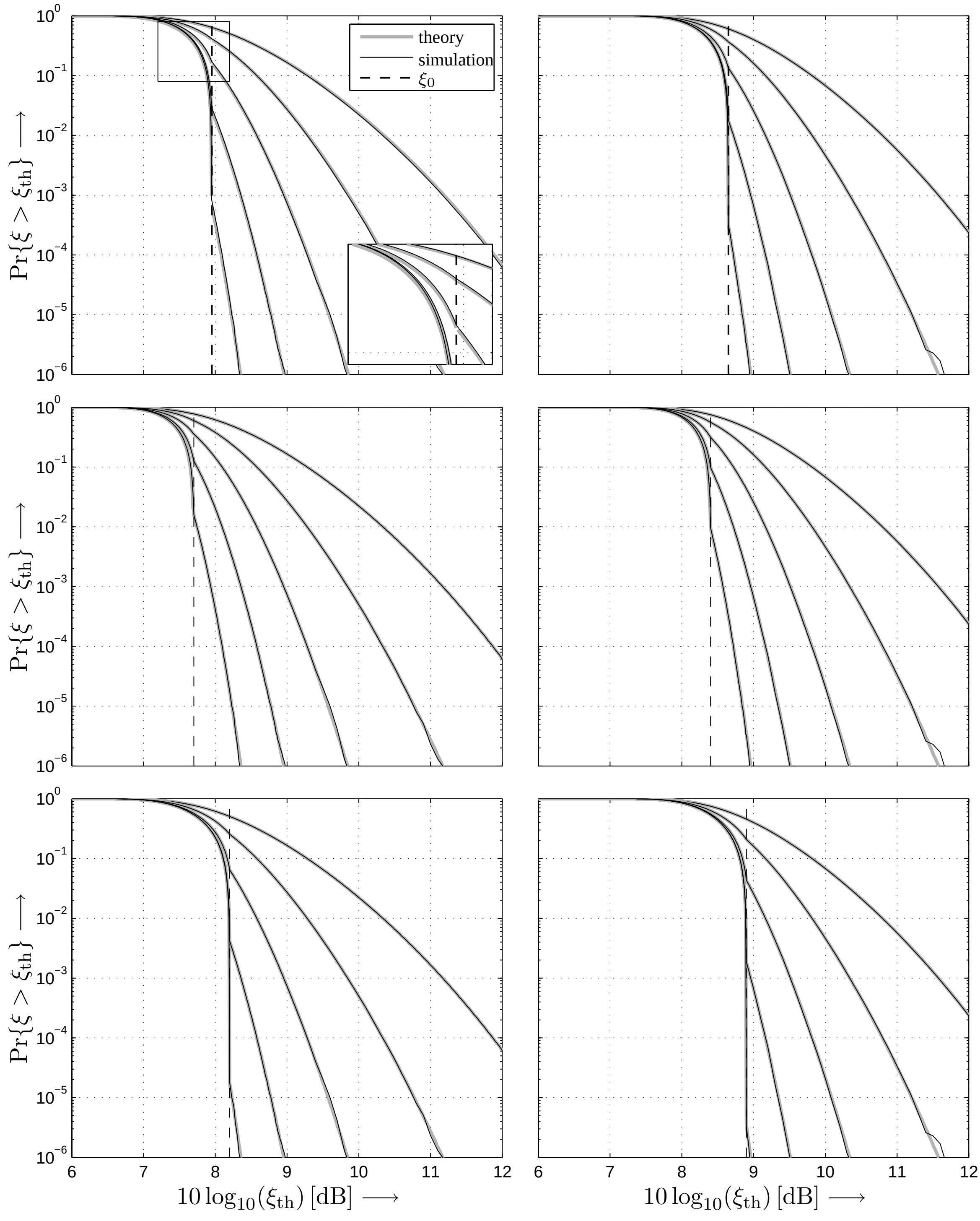


Fig. 2. Ccdf of $\xi_{\rm th}$ for SSLM with (from left to right) $\hat{U} = 16, 8, 4, 2, 1$; theoretic results (top left subfigure: based on Gaussian signalling; other subfigures: based on numericals results of original signal) are depicted gray; numerical simulations black; left column: no oversampling ($L = 1$); right column: eight times oversampling ($L = 8$); top row: threshold $\xi_0 = \xi_\infty$ (left: $\xi_0 \mathrel{\widehat{=}} 7.95\,\mathrm{dB}$, right: $\xi_0 \mathrel{\widehat{=}} 8.65\,\mathrm{dB}$); middle row: threshold $10\log_{10}(\xi_0) = 10\log_{10}(\xi_\infty) - 0.25\,\mathrm{dB}$ (left: $\xi_0 \mathrel{\widehat{=}} 7.7\,\mathrm{dB}$, right: $\xi_0 \mathrel{\widehat{=}} 8.4\,\mathrm{dB}$); bottom row: threshold $10\log_{10}(\xi_0) = 10\log_{10}(\xi_\infty) + 0.25\,\mathrm{dB}$ (left: $\xi_0 \mathrel{\widehat{=}} 8.2\,\mathrm{dB}$, right: $\xi_0 \mathrel{\widehat{=}} 8.9\,\mathrm{dB}$); $D = 512$, ($M = 4$)-QAM signaling per carrier.

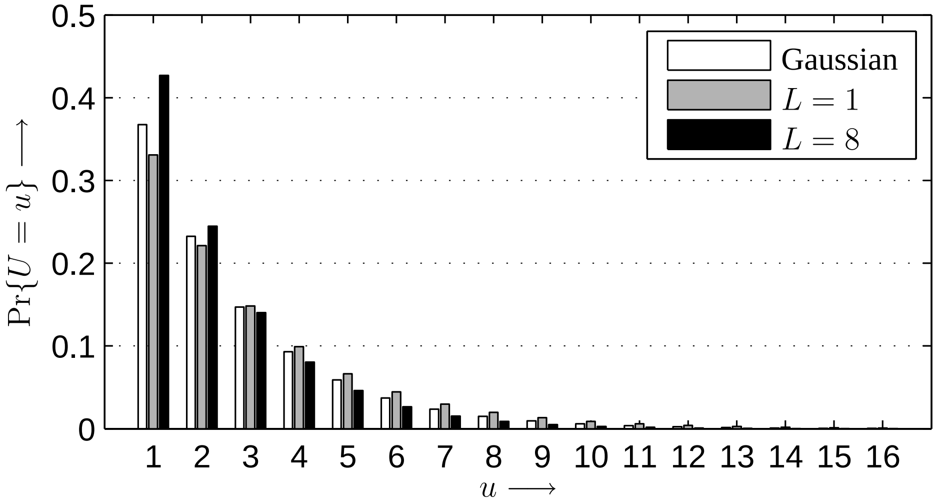
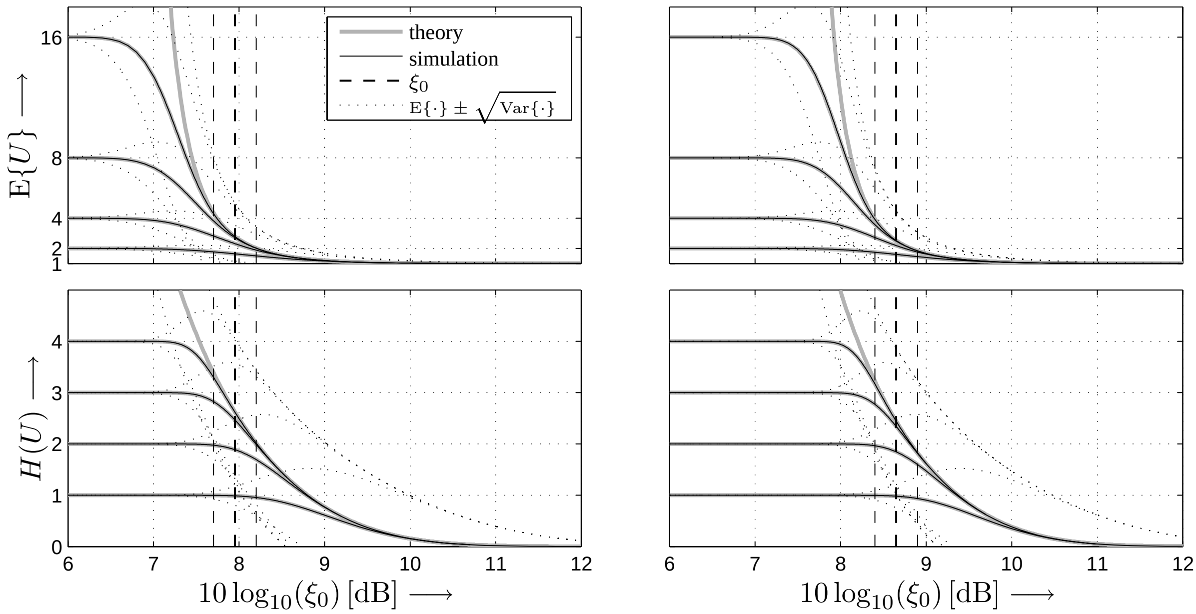


Fig. 3. Probabilities $\Pr\{U = u\}$ that u according to (15); left: Gaussian signalling; middle: no oversampling ($L = 1$); right: eight times oversampling ($L = 8$); $D = 512$, ($M = 4$)-QAM signaling per carrier.



$\mathrm{E}\{U\}$		$10\log_{10}(\xi_0)$ 7.7 dB	7.95 dB	8.2 dB
$\hat{U}$	$\to\infty$	4.38	2.79	2.02
	16	4.31	2.79	2.02
	8	3.83	2.71	2.01
	4	2.83	2.32	1.89
	2	1.77	1.64	1.50

$\mathrm{E}\{U\}$		$10\log_{10}(\xi_0)$ 8.4 dB	8.65 dB	8.9 dB
$\hat{U}$	$\to\infty$	3.96	2.52	1.84
	16	3.92	2.52	1.84
	8	3.58	2.48	1.83
	4	2.72	2.19	1.76
	2	1.75	1.60	1.46

$H(U)$		$10\log_{10}(\xi_0)$ 7.7 dB	7.95 dB	8.2 dB
$\hat{U}$	$\to\infty$	3.39	2.63	2.02
	16	3.30	2.62	2.02
	8	2.83	2.47	1.99
	4	1.98	1.89	1.69
	2	1.00	0.99	0.96

$H(U)$		$10\log_{10}(\xi_0)$ 8.4 dB	8.65 dB	8.9 dB
$\hat{U}$	$\to\infty$	3.23	2.44	1.83
	16	3.17	2.44	1.83
	8	2.77	2.34	1.81
	4	1.97	1.84	1.60
	2	1.00	0.98	0.94

Fig. 4. Top: Average number of assessed signal candidates $\mathrm{E}\{U\}$ (top row) and average number of required bits of side information $H(U)$ (bottom row) over thresholds ξ_0 with (from top to bottom) $\hat{U} = \infty, 16, 8, 4, 2$; theoretic results are depicted gray; numerical simulations black; left column: no oversampling ($L = 1$); right column: eight times oversampling ($L = 8$); dotted lines represent $\mathrm{E}\{U\} \pm \sqrt{\mathrm{Var}\{U\}}$ or $H(U) \pm \sqrt{\mathrm{Var}\{-\log_2(\Pr\{U = u\})\}}$; dashed lines mark the respective thresholds ξ_0 chosen in Fig. 2; $D = 512$, ($M = 4$)-QAM signaling per carrier. Bottom: Resulting values of $\mathrm{E}\{U\}$ and $H(U)$ at the marked thresholds ξ_0.

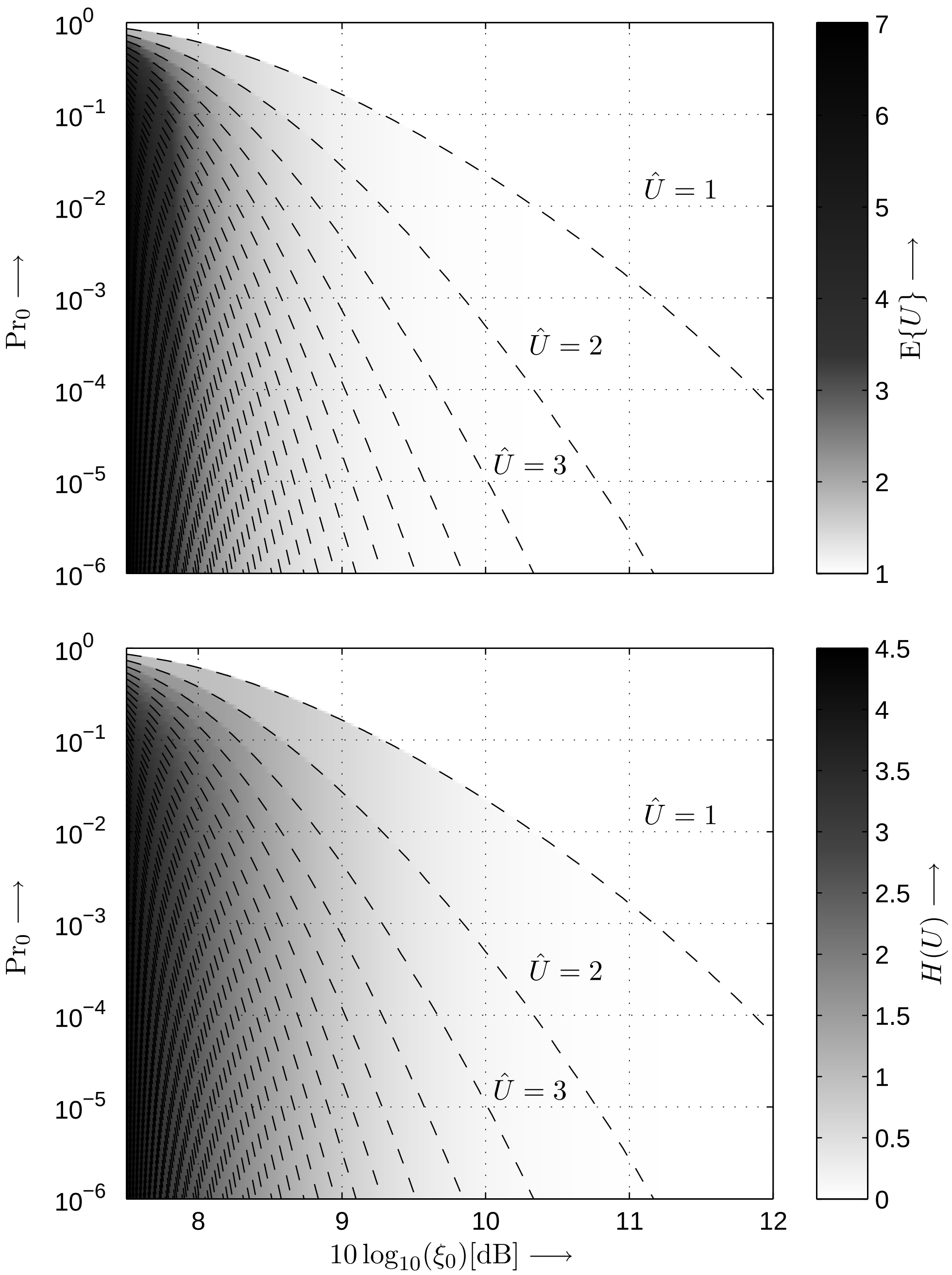


Fig. 5. Contour plot of achievable values of the expected value $\mathrm{E}\{U\}$ (top) and the entropy $H(U)$ (bottom) for different design parameters (ξ_0, Pr_0); regions for different resulting values of $\hat{U}$ (according to (32)) are delimited by dashed lines; $D = 512$.

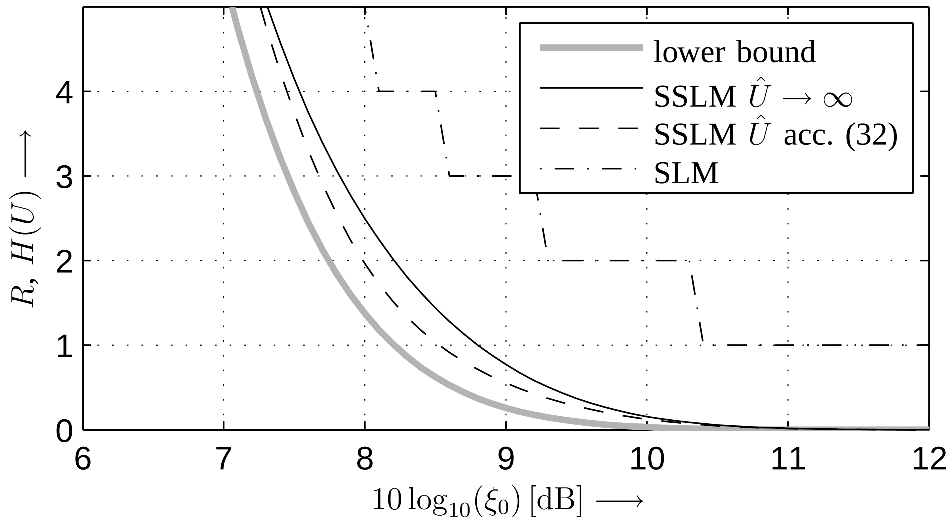


Fig. 6. Required redundancy bits according to lower bound [28], [27], SSLM with $\hat{U} \to \infty$, SSLM according to Sec. IV-D, and pure SLM (the later two are considered at a clipping probability of $\mathrm{Pr}_0 = 10^{-4}$; $D = 512$, $(M = 4)$-QAM signaling per carrier.