Optimum Nyquist Windowing in OFDM Receivers

Stefan H. Müller-Weinfurtner

Abstract—**Conventional orthogonal frequency-division multiplexing receivers disregard the guard interval in channels with small dispersion even though its unconsumed portion can be exploited for improved demodulation by applying a Nyquist-shaped window in the receiver. The transmitter does not need to be modified, subcarrier orthogonality in the receiver is preserved, and the receiver discrete Fourier transform size can be retained. Windowing can mitigate the joint effect of additive noise and intercarrier interference among subcarriers caused by the carrier frequency offset. In this letter, a closed solution for optimum window coefficients suited to this scenario is derived.**

Index Terms—**Carrier frequency offset, multicarrier transmission, Nyquist window, OFDM, optimum demodulation.**

I. Introduction

CONVENTIONAL orthogonal frequency-division multiplexing (OFDM) reception [1] ignores the guard interval (cyclic prefix and/or postfix). Only the so-called useful samples are fed to a discrete Fourier transform (DFT) which performs conventional OFDM demodulation, being equivalent with rectangular windowing of the received signal. Imperfections like residual frequency offset, oscillator phase noise, and Doppler spread cause intercarrier interference (ICI) among subcarriers, and we desire to lower the sensitivity of OFDM to frequency offsets.

In many situations, the specified guard interval is oversized, i.e., the actual duration of the channel-impulse response is shorter than the fixed guard interval. We refer to that part of the guard interval which is not affected by channel echoes originating from previous OFDM symbols as *unconsumed guard interval*. Samples in that region are exploitable to mitigate additive noise and ICI from residual carrier frequency offsets. The achievable windowing gains are not tremendous, but obtainable by simple preprocessing prior to the conventional-sized DFT. In [2], a raised-cosine windowing is used, and in [3], an optimization of the Nyquist shape has been performed to minimize additive noise power in the subcarriers.

Paper approved by A. Ahlen, the Editor for Modulation and Signal Design of the IEEE Communications Society. Manuscript received June 9, 1999; revised August 31, 1999 and June 6, 2000. This work was supported in part by Ericsson Eurolab Deutschland GmbH in Nürnberg and in part by the research initiative ATMmobil of the German Ministry for Research and Education (BMBF). This paper was presented in part at the 50th IEEE Vehicular Technology Conference (VTC 1999–Fall), Amsterdam, The Netherlands, September 1999, and at the IEEE Global Telecommunications Conference (GLOBECOM 2000), San Francisco, CA, November 2000.

The author was with Lehrstuhl für Nachrichtentechnik II, Universität Erlangen-Nürnberg, D-91058 Erlangen, Germany. He is now with AT&T Laboratories–Research, Middletown, NJ 07748 USA (e-mail: smw@research.att.com).

Publisher Item Identifier S 0090-6778(01)02169-9.

II. Transmission Model

A. Conventional OFDM Transmitter

The OFDM symbol is generated by a D-point inverse DFT. Subcarrier ν is modulated by the complex amplitude A_ν from the same zero-mean signal set for all ν. The modulation interval is T and the subcarrier spacing is $\Delta f_{\rm sub} = 1/DT$. A guard interval [1] with $D_{\rm g} < D$ samples in total is added. It is split up into a cyclic prefix and a cyclic postfix of $D_{\rm pr}$ and $D_{\rm po}$ samples, respectively. The discrete-time complex baseband transmit samples are $s_k = (1/\sqrt{D})\sum_{\nu=0}^{D-1} A_\nu {\rm e}^{+{\rm j}(2\pi/D)\nu k}$, $-D_{\rm pr} \le k < D+D_{\rm po}$, where k is the discrete time. The average signal power is $\sigma_s^2 \triangleq \mathcal{E}\{|s_k|^2\}$.

B. OFDM Receiver with Nyquist Window

A nondispersive channel with residual carrier frequency offset $\Delta f_{\rm co}$ is considered; the noiseless receive sample is $\tilde{r}_k = {\rm e}^{+{\rm j}2\pi\Delta f_{\rm co}Tk} s_k$ with $\mathcal{E}\{|\tilde{r}_k|^2\} = \sigma_s^2$. Zero-mean additive white Gaussian noise with $\sigma_n^2 \triangleq \mathcal{E}\{|n_k|^2\}$ is added such that $r_k = \tilde{r}_k + n_k$. The channel signal-to-noise power ratio (SNR) is $\varsigma_{\rm c} \triangleq \sigma_s^2/\sigma_n^2$. The modified OFDM demodulation is performed by

$$Y_\nu = \frac{1}{\sqrt{D}} \sum_{k=-\infty}^{+\infty} w_k r_k {\rm e}^{-{\rm j}(2\pi/D)\nu k} = \frac{1}{\sqrt{D}} \sum_{\kappa=0}^{D-1} \underbrace{\left(\sum_{n=-1}^{+1} w_{nD+\kappa} r_{nD+\kappa} \right)}_{\triangleq y_\kappa} {\rm e}^{-{\rm j}(2\pi/D)\nu\kappa} \quad (1)$$

where we substituted $k = \kappa + nD$. The modification consists in windowing r_k with w_k and for validity of the summation limits in (1), we require $w_k = 0$ at least for $k < -D$ and $k \ge 2D$. Now, y_κ and not r_κ is input to the DFT of size D. The efficient implementation in (1) is related with the polyphase implementation of a decimated DFT filter bank [4, p. 127] and is in contrast to [2], where (1) is directly calculated with a DFT of size $2D$. Introducing the normalized frequency offset (NFO) $\xi_f \triangleq \Delta f_{\rm co}/\Delta f_{\rm sub} = \Delta f_{\rm co} TD$, $|\xi_f| < 0.5$, the noiseless frequency-domain transmission characteristic $Y_\nu = \sum_{\nu_{\rm t}=0}^{D-1} A_{\nu_{\rm t}} H_{\xi_f}[\nu_{\rm t} - \nu]$ is obtained, with window-dependent *transfer factor* (TF)

$$H_{\xi_f}[\Delta\nu] \triangleq \frac{1}{D} \sum_{k=-\infty}^{+\infty} w_k {\rm e}^{+{\rm j}(2\pi/D)(\Delta\nu+\xi_f)k} \quad (2)$$

characterizing the crosstalk between any $\Delta\nu$-spaced transmitted subcarrier $\nu_{\rm t} = \nu + \Delta\nu$ and received subcarrier ν for given NFO

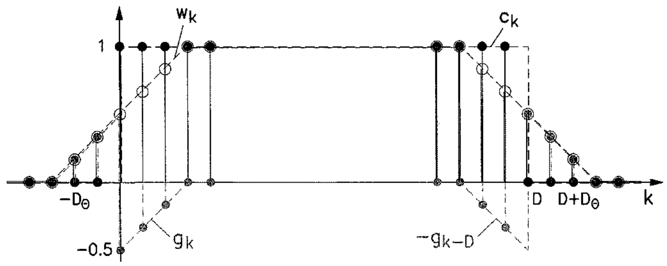

Fig. 1. Illustrative decomposition of a Nyquist receiver time window w_k into the sum of a rectangular and two roll-off modifications with $D_\theta = 2$.

and (cyclic) subcarrier index spacing $\Delta\nu \in \mathbb{Z}$. To ensure subcarrier *orthogonality* for $\xi_f = 0$, the time window must comply with the time-domain *Nyquist condition*

$$\sum_{i=-\infty}^{+\infty} w_{k+iD} = \text{const}, \qquad 0 \le k < D. \tag{3}$$

A second (practical) demand is the finite duration of the window shape.

III. MMSE Optimum Nyquist Window Shape

A. Decomposition of Nyquist Window

The rectangular window component

$$c_k = \begin{cases} 1, & 0 \le k < D \\ 0, & k < 0 \text{ or } k \ge D \end{cases}$$

from conventional OFDM is appropriately modified [5], [6] by the time-limited roll-off modification g_k with $g_k = 0, \forall |k| > D_\theta$. The time-excess parameter $0 \le D_\theta < D/2$ is variable. The window composition is

$$w_k = c_k + g_k - g_{k-D} \tag{4}$$

as this structure enforces (3) without further restriction. Please refer to Fig. 1 for this decomposition into a rectangular and two roll-off components. The window is extended by D_θ samples on the left and $D_\theta + 1$ samples on the right, when compared to c_k, yielding the time-excess factor $\theta = (2D_\theta + 1)/D$.

B. MMSE Optimization of Window Shape

With (4) and further manipulations [5], the TF from (2) can be rewritten as

$$H_{\xi_f}[\Delta\nu] = \frac{\mathrm{e}^{+\mathrm{j}2\pi\xi_f} - 1}{D} \cdot \left(\frac{1}{\mathrm{e}^{+\mathrm{j}(2\pi/D)(\Delta\nu+\xi_f)} - 1} - \sum_{k=-D_\theta}^{+D_\theta} \mathrm{e}^{+\mathrm{j}(2\pi/D)(\Delta\nu+\xi_f)k} g_k \right). \tag{5}$$

The noiseless and interference-free received amplitude in subcarrier ν is $A_\nu H_{\xi_f}[0]$. The average power of ICI in one specific subcarrier induced from all other active subcarriers is σ_I^2. Given all subcarriers are operated with identical average power, then σ_I^2 is identical for all subcarriers. The average noise power per subcarrier is denoted by σ_N^2. As both are assumed to be uncorrelated, the overall interference and noise power in one subcarrier is given by $\sigma_I^2 + \sigma_N^2$ and this mean-squared error (MSE) is minimized by optimizing the g_k's. As the Nyquist criterion is already enforced by the construction principle, we have $H_{\xi_f}[0] \approx 1$ under the constraint that ξ_f is not too large. Consequently, the roll-off shape which minimizes $\sigma_I^2 + \sigma_N^2$ will in good approximation maximize the subcarrier SNR

$$\varsigma_\mathrm{s} \triangleq \sigma_s^2 |H_{\xi_f}[0]|^2 / (\sigma_I^2 + \sigma_N^2). \tag{6}$$

If all subcarriers are used, then the subcarrier SNR for a window with θ is upper bounded by [3]

$$\varsigma_\mathrm{s} \le \frac{2}{2-\theta} \varsigma_\mathrm{c}. \tag{7}$$

The average ICI power in each subcarrier is[1] $\sigma_I^2 = \sigma_s^2 \boldsymbol{H}^\mathrm{H} \boldsymbol{H}$ with $\boldsymbol{H} \triangleq [H_{\xi_f}[1], \ldots, H_{\xi_f}[D-1]]^\mathrm{T}$ (vector of interfering TFs). The additive noise in the subcarriers is acquired by windowing of white time-domain noise samples and its average power per subcarrier is $\sigma_N^2 = (1/D)\sigma_n^2 \boldsymbol{w}^\mathrm{H}\boldsymbol{w}$ with window coefficient vector $\boldsymbol{w} \triangleq [w_{-D_\theta}, \ldots, w_{D+D_\theta}]^\mathrm{T}$ of length $D + 2D_\theta + 1$. At this point, we introduce

$$\boldsymbol{a} = \left[\frac{1}{\mathrm{e}^{+\mathrm{j}(2\pi/D)(1+\xi_f)} - 1}, \ldots, \frac{1}{\mathrm{e}^{+\mathrm{j}(2\pi/D)((D-1)+\xi_f)} - 1} \right]^\mathrm{T} \tag{8}$$

$$\boldsymbol{B} = \left[\mathrm{e}^{+\mathrm{j}(2\pi/D)(\Delta\nu+\xi_f)k} \right]_{\substack{\Delta\nu=1,\ldots,D-1 \\ k=-D_\theta,\ldots,D_\theta}} \tag{9}$$

$$\boldsymbol{g} = [g_{-D_\theta}, \ldots, g_{D_\theta}]^\mathrm{T} \tag{10}$$

$$\boldsymbol{d} = [\underbrace{0, \ldots, 0}_{D_\theta}, \underbrace{1, \ldots, 1}_{D}, \underbrace{0, \ldots, 0}_{D_\theta+1}]^\mathrm{T} \tag{11}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{I}_{(2D_\theta+1)\times(2D_\theta+1)} \\ \boldsymbol{O}_{(D-2D_\theta-1)\times(2D_\theta+1)} \\ -\boldsymbol{I}_{(2D_\theta+1)\times(2D_\theta+1)} \end{bmatrix}. \tag{12}$$

Matrix $\boldsymbol{B}$ has $D-1$ rows and $2D_\theta + 1$ columns. Vector $\boldsymbol{g}$ is the unknown roll-off modification of length $2D_\theta + 1$, which is to be optimized. Vector $\boldsymbol{d}$ is the rectangular window of conventional OFDM, modified by the additive roll-off contribution $\boldsymbol{Cg}$. $\boldsymbol{I}_{a\times a}$ and $\boldsymbol{O}_{a\times b}$ represent the identity and zero matrix of given sizes, respectively. Now, we can rewrite (5) compactly as $\boldsymbol{H} = ((\mathrm{e}^{+\mathrm{j}2\pi\xi_f} - 1)/D)(\boldsymbol{a} - \boldsymbol{Bg})$ and the window composition from (4) as $\boldsymbol{w} = \boldsymbol{d} + \boldsymbol{Cg}$. For optimization, the minimum MSE (MMSE) approach is chosen. We set the derivative $\partial(\sigma_I^2 + \sigma_N^2)/\partial\boldsymbol{g}$ to zero and obtain the optimum SNR-adaptive roll-off modification[2]

$$\boldsymbol{g} = \left(\frac{\varsigma_\mathrm{c}}{D} |\mathrm{e}^{+\mathrm{j}2\pi\xi_f} - 1|^2 \Re\{\boldsymbol{B}^\mathrm{H}\boldsymbol{B}\} + 2\boldsymbol{I} \right)^{-1} \cdot \left(\frac{\varsigma_\mathrm{c}}{D} |\mathrm{e}^{+\mathrm{j}2\pi\xi_f} - 1|^2 \Re\{\boldsymbol{B}^\mathrm{H}\boldsymbol{a}\} - \boldsymbol{C}^\mathrm{T}\boldsymbol{d} \right). \tag{13}$$

C. Asymptotic Optimum Results

1) For channel SNRs $\varsigma_\mathrm{c} \rightarrow \infty$ (∞ dB), i.e., dominating ICI, we obtain from (13) the asymptotic result

[1] $\boldsymbol{X}^\mathrm{H}$ denotes conjugate transposition (Hermitian transposition) and $\boldsymbol{X}^\mathrm{T}$ transposition of the matrix/vector $\boldsymbol{X}$.

[2] It can be easily shown that $\boldsymbol{C}^\mathrm{T}\boldsymbol{C} = 2\boldsymbol{I}_{(2D_\theta+1)\times(2D_\theta+1)}$.

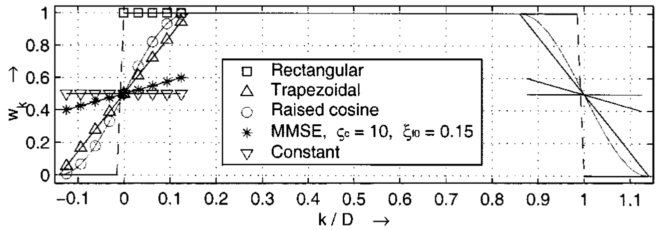

Fig. 2. Illustration of various Nyquist roll-off shapes $\boldsymbol{w}$ with $D_\theta = 8$ and $D = 64$ so that the window has a time-excess factor of $\theta = 0.266$.

$\boldsymbol{g} = (\Re\{\boldsymbol{B}^{\mathrm{H}}\boldsymbol{B}\})^{-1}\Re\{\boldsymbol{B}^{\mathrm{H}}\boldsymbol{a}\}$. The result only depends on the NFO.

2) Another interesting case is observed for $\xi_f = 0$ or at extremely low SNRs, where we have $\mathrm{e}^{+\mathrm{j}2\pi\xi_f} = 1$ or $\varsigma_{\mathrm{c}} \to 0\,(-\infty$ dB), respectively. In this case, additive noise dominates and the constant solution

$$\boldsymbol{g} = -(2\boldsymbol{I})^{-1}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{d} = [\underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{D_\theta}, \underbrace{-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}}_{D_\theta+1}]^{\mathrm{T}}$$

is obtained. The resulting window is constantly 0.5 in both roll-off regions [3], which can basically be interpreted as optimum combining rule for all those signal samples which are received twice due to the unconsumed guard interval. We refer to this cheaply implementable window as *constant* Nyquist shape.

D. Numerical Results for MMSE Optimum Window

Fig. 2 shows $\boldsymbol{w}$ for $D = 64$ with $\theta = 0.266$, obtained from (13) for $\xi_f = 0.15$ and $\varsigma_{\mathrm{c}} = 10$ (10 dB) ($\to$ MMSE, $*$), and for $\varsigma_{\mathrm{c}} \to 0$ ($-\infty$ dB) or $\xi_f = 0$ ($\to$ Constant, $\bigtriangledown$), which is constantly 0.5 in the roll-off region. To the left and to the right, we find discontinuities of the optimum window shapes. Both optimum solutions exhibit an approximately linear course inside the roll-off region, which led to the simplified derivation in [7] with *a priori* linear roll-off segment. There, a much simpler near-optimum solution is obtained which requires no matrix inversion.

IV. Performance Comparison with Other Windows

The three other window shapes in Fig. 2 are now compared to the constant and the MMSE optimum shape. We investigate the rectangular shape of conventional OFDM, a Nyquist shape with trapezoidal and raised-cosine shape [2]. All Nyquist windows have the time-excess factor $\theta = 0.266$. Possible gains in joint noise and ICI power are demonstrated in Fig. 3, where the subcarrier SNR ς_{s} is plotted over the NFO ξ_f for all five windows from Fig. 2. The channel SNR is constantly $10\log_{10}\varsigma_{\mathrm{c}} = 10$ dB. From Fig. 3 at $\xi_f = 0$, we see that a maximum signal combining gain of [cf. (7)] 0.62 dB results for $\theta = 0.266$ compared to conventional OFDM which cannot exceed the channel SNR of 10 dB. The constant shape achieves the maximum gain at $\xi_f = 0$. The MMSE optimum shape shows superior behavior with nonzero NFOs, while the degradation at $\xi_f = 0$ is hardly visible. The diagram can also be read as follows: conventional OFDM drops to an effective subcarrier SNR of 9.5 dB at $\xi_f = 0.06$, while twice that offset is tolerable with windowing.

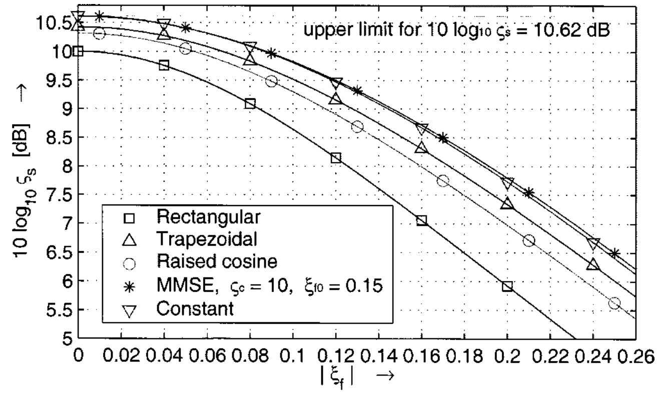

Fig. 3. Subcarrier SNR versus NFO for various Nyquist roll-off shapes with $D_\theta = 8$ at $D = 64$, i.e., $\theta = 0.266$. A channel SNR of $10\log_{10}\varsigma_{\mathrm{c}} = 10$ dB is considered.

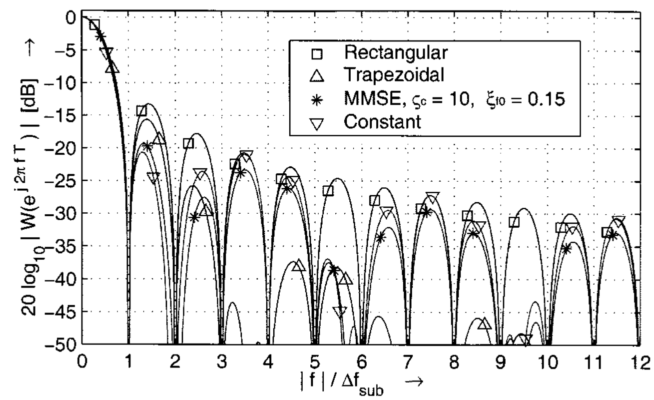

Fig. 4. Effective frequency responses for subcarrier demodulation with four different receiver window shapes with $D_\theta = 8$ and $D = 64$, i.e., $\theta = 0.266$.

For small θ, the biggest portion of the gain is due to the signal combining gain, which in good approximation is fully realized by the simple constant Nyquist window. The performance advantage of the MMSE optimum window over the constant solution increases for larger θ, so that there the general solution is required for optimality. For $\theta \to 1$, the MMSE solution approaches a triangular window shape. The raised-cosine roll-off region [2] is globally suboptimum; it offers only little more than half of the possible gains. Even a trapezoidal Nyquist shape achieves a better effective subcarrier SNR.

As frequency-domain explanation for optimality, we investigate the normalized frequency response

$$\begin{aligned} W(\mathrm{e}^{+\mathrm{j}2\pi fT}) &\triangleq \frac{1}{D}\sum_{k=-\infty}^{+\infty} w_k \mathrm{e}^{-\mathrm{j}2\pi fTk} \\ &= \frac{1}{D}\sum_{k=-\infty}^{+\infty} w_k \mathrm{e}^{-\mathrm{j}(2\pi/D)(f/\Delta f_{\mathrm{sub}})k} \end{aligned} \tag{14}$$

of four receive window shapes $\boldsymbol{w}$ with $\theta = 0.266$ in Fig. 4. All Nyquist-type time windows offer reduced side lobes when compared to original OFDM. The MMSE optimized solutions have significantly reduced side lobes directly adjacent to the main lobe (see between subcarrier index 1 and 3), when compared to the trapezoidal window. It is this feature, which decisively decreases interference power in case of residual carrier frequency offsets. The trapezoidal shape without time-domain discontinuities results in a faster-decreasing frequency response compared to the ones of MMSE optimum window solutions with discontinuities.

V. Summary and Conclusions

We exploited an oversized guard interval in the receiver to cheaply mitigate joint additive noise and ICI due to carrier frequency offsets. We derived the MMSE optimum Nyquist window and compared its performance to that of known windows. Achievable gains are not that large, but the intuitive raised-cosine shape is suboptimum. Further, the shape from [3] is obtained as asymptotic solution.

Our presentation assumed an OFDM system with pre- and postfix which is not compulsory for windowing. With introduction of an appropriate time shift, each existing OFDM receiver can be modified to perform Nyquist windowing.

Acknowledgment

The author would like to thank Prof. Dr.-Ing. J. B. Huber and Dipl.-Ing. Univ. G. Bumiller for fruitful discussions.

References

[1] H. Sari, G. Karam, and I. Jeanclaude, "Transmission techniques for digital terrestrial TV broadcasting," *IEEE Commun. Mag.*, vol. 33, pp. 100–109, Feb. 1995.

[2] C. Muschallik, "Improving an OFDM reception using an adaptive Nyquist windowing," *IEEE Trans. Consumer Electron.*, vol. 42, no. 3, pp. 259–269, 1996.

[3] R. Klöckner, "Nutzung des Schutzintervalls bei OFDM zur Optimierung des Signal-Rausch-Abstands" (in German), in *3. OFDM-Fachgespräch*, Braunschweig, Germany, September 1998.

[4] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[5] S. Müller-Weinfurtner, "OFDM for wireless communications: Nyquist windowing, peak-power reduction, and synchronization," in *Reihe Kommunikations und Informationstechnik, Band 16*: Shaker Verlag, Aachen, Dissertation Universität Erlangen-Nürnberg, 2000.

[6] S. Müller-Weinfurtner and J. Huber, "Optimum Nyquist windowing for improved OFDM receivers," in *Proc. IEEE Global Telecommunications Conf. (GLOBECOM 2000)*, San Francisco, CA, Nov. 2000, pp. 711–715.

[7] ——, "Robust OFDM reception with near-optimum Nyquist window," in *Proc. 50th IEEE Vehicular Technology Conf. (VTC 1999-Fall)*, Amsterdam, The Netherlands, Sept. 1999, pp. 289–293.