Performance of Peak-to-Average Power Ratio Reduction in Single- and Multi-Antenna OFDM via Directed Selected Mapping

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Abstract

Selected mapping (SLM) is a popular scheme for peak power reduction in orthogonal frequency-division multiplexing (OFDM) systems. In this letter, the performance of various versions of SLM, among them ordinary and directed SLM, in single- and multi-antenna point-to-point OFDM systems is assessed. Analytic expressions for the distribution of the PAR are derived. Numerical results cover that significant gains over conventional SLM can be achieved by directed SLM.

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I. Introduction

It is well known that the orthogonal frequency-division multiplexing (OFDM) transmit signal is almost Gaussian distributed and hence exhibits a large *peak-to-average power ratio (PAR)* [3]. Clipping of these peaks by non-linear amplifiers will cause undesirable out-of-band radiation and hence violation of spectral masks. In order not to operate with large power back-offs, an algorithmic control of the transmit signal for reducing PAR is required.

Over the last decade, a variety of PAR reduction techniques for OFDM has been developed [5]. Among them, *redundant signal representations*, in particular *selected mapping (SLM)* [2] is of special interest. Recently, SLM has been extended to multiple-input/multiple-output (MIMO) OFDM [6], [1], [4]. However, an in-depth study of the performance and in particular the derivation of the achievable gains over single-antenna (single-input/singe-output, SISO) transmission and the characterization whether the MIMO PAR reduction scheme is able to exploit the inherent potential of MIMO over SISO transmission are still missing.

In this paper, we assess SLM in point-to-point MIMO OFDM. Thereby the simple application of SISO SLM in parallel [1] is compared with *directed SLM (dSLM)* [4], [9] which is tailored to the MIMO situation. Using this technique, similar gains as in error performance of MIMO system (achieving diversity gain, i.e., increasing the slope of the error rate curve over the signal-to-noise ratio) can be observed for PAR reduction as well. In addition, it is shown, that constituting frames in temporal direction (we call them *hyper frames*), the principle of dSLM can be applied to SISO schemes as well.

In Section II, the system model is introduced. The performance analysis of the various SLM schemes, both for SISO and MIMO systems, is given in Section III. The PAR reduction schemes are assessed via numerical simulations in Section IV, and Section V draws some conclusions.

II. OFDM System Model

We consider a discrete-time OFDM system model [3], employing an (inverse) discrete Fourier transform ((I)DFT) of length D; all D carriers are expected to be active. Moreover, we assume N_T transmit antennas, over which mutually independent data streams are communicated; space-time coding (resulting in dependent signals) is not considered.

In each of the N_T parallel OFDM transmitters binary data is mapped to complex-valued data symbols $A_{\mu,d}$, $\mu = 1, \ldots, N_\mathrm{T}$, $d = 0, \ldots, D-1$, taken from an M-ary QAM or PSK constellation with variance σ_a^2. The frequency-domain OFDM frames—collecting the D data symbols of each antenna and denoted by $\boldsymbol{A}_\mu = [A_{\mu,0}, \ldots, A_{\mu,D-1}]$—are transformed into the time-domain OFDM frames $\boldsymbol{a}_\mu = [a_{\mu,0}, \ldots, a_{\mu,D-1}]$ via IDFT. The correspondence is written in short as $\boldsymbol{a}_\mu = \mathrm{IDFT}\{\boldsymbol{A}_\mu\}$.

A. Peak-to-Average Power Ratio

Because of the statistical independence of the carriers, the time-domain samples $a_{\mu,k}$ are approximately complex Gaussian distributed.[1] This results in a high peak-to-average power ratio $\xi_\mu \overset{\text{def}}{=} \max_k |a_{\mu,k}|^2/\mathrm{E}\{|a_{\mu,k}|^2\}$. To avoid non-linear distortion in the power amplifiers and in turn the generation of undesired out-of-band radiation, the PAR of all N_T transmit signals should be simultaneously as small as possible. Since performance is limited by the worst-case PAR,[2] (the other simultaneously transmitted frames are masked) we consider

$$\xi \overset{\text{def}}{=} \max_{\mu=1,\ldots,N_\mathrm{T}} \xi_\mu = \max_{\mu,k} \left(|a_{\mu,k}|^2/\mathrm{E}\{|a_{\mu,k}|^2\}\right) . \tag{1}$$

As performance measure for PAR reduction we study the probability that the PAR of an OFDM frame exceeds a given threshold ξ_th, i.e., the *complementary cumulative distribution function (ccdf)* of PAR $\mathrm{ccdf}(\xi_\mathrm{th}) \overset{\text{def}}{=} \Pr\{\xi > \xi_\mathrm{th}\}$ ($\Pr\{\cdot\}$: probability).

Reference is the ccdf of conventional SISO OFDM without any PAR reduction technique, $\mathrm{ccdf}_\mathsf{SISO}(\xi_\mathrm{th})$. Noteworthy, assuming Gaussian time-domain samples at Nyquist rate this ccdf can be well approximated by [2]

$$\mathrm{ccdf}_\mathsf{SISO}(\xi_\mathrm{th}) \overset{\text{Gauss}}{=} 1 - (1 - \mathrm{e}^{-\xi_\mathrm{th}})^D \tag{2}$$

In MIMO OFDM, since $N_\mathrm{T}D$ instead of D time-domain samples are present, the same situation as in SISO OFDM with $N_\mathrm{T}D$ carriers results, and $\mathrm{ccdf}_\mathsf{MIMO}(\xi_\mathrm{th}) \overset{\text{Gauss}}{=} 1 - (1 - \mathrm{e}^{-\xi_\mathrm{th}})^{N_\mathrm{T}D}$ holds, which is even worse than (2).

III. Performance of Conventional and Directed SLM

A. Original (Single-Antenna) Selected Mapping

In SLM each OFDM frame is *mapped* to a number of U (independent) candidates representing the same information. From these the one with the lowest PAR (or any other criteria) is *selected*. These candidates are obtained by using U different *mappings* $\mathcal{M}^{(u)}$,

[1] We concentrate on the PAR of the discrete-time samples $a_{\mu,k}$. By replacing the respective original ccdf the derived performance comparison is also valid for the situation when using an oversampled IDFT.

[2] Other criteria as the *harmonic mean* of the PAR [8] may also be taken into account.

$u = 1, \dots, U$, i.e., bijective transformations of a frequency-domain vector $\boldsymbol{A}$ onto alternative versions $\boldsymbol{A}^{(u)} = \mathcal{M}^{(u)}(\boldsymbol{A})$. Mostly, the mapping is implemented as carrier-wise multiplication of $\boldsymbol{A}$ by U phase vectors $\boldsymbol{P}^{(u)} \stackrel{\text{def}}{=} [P_0^{(u)}, \dots, P_{D-1}^{(u)}]$, $u = 1, \dots, U$, $|P_d^{(u)}| = 1$, randomly selected and known to transmitter and receiver. Other mappings, e.g., by permuting the elements of $\boldsymbol{A}$ or via scrambling (linear filtering over the binary field) are possible, too.

The candidates are transformed to time domain, $\boldsymbol{a}^{(u)} = \mathrm{IDFT}\{\mathcal{M}^{(u)}(\boldsymbol{A})\}$ (U IDFTs are required), their PARs are calculated, and the "best" OFDM frame $\boldsymbol{a}^{(u^*)}$ is actually transmitted. In order to recover data side information bits (the index u^*) to indicate the mapping $\mathcal{M}^{(u^*)}$ have to be communicated to the receiver.

Using SLM the ccdf (and assuming Gaussian time-domain samples) is given by [2]

$$\mathrm{ccdf}_{\mathsf{SLM}}(\xi_{\mathrm{th}}) = (\mathrm{ccdf}_{\mathsf{SISO}}(\xi_{\mathrm{th}}))^U \stackrel{\text{Gauss}}{=} (1 - (1 - \mathrm{e}^{-\xi_{\mathrm{th}}})^D)^U \approx D^U \cdot \mathrm{e}^{-\xi_{\mathrm{th}} U} \,, \qquad (3)$$

where for the last approximation $(1 - \mathrm{e}^{-x})^y \approx 1 - y\mathrm{e}^{-x}$ has been used, which becomes exact as $x \to \infty$.

B. *Ordinary SLM for MIMO OFDM*

In *ordinary SLM (oSLM)* [1] SISO SLM is applied N_{T} times in parallel. For each of the parallel OFDM frames the best mapping out of the U possible is individually selected. Here, $N_{\mathrm{T}}U$ IDFTs are required (the same complexity per antenna as in SISO SLM).

Since the worst-case PAR (1) does not exceed ξ_{th} if all individual PARs stay below, and the individual PARs are distributed according to (3), we straightforwardly obtain

$$\mathrm{ccdf}_{\mathsf{oSLM}}(\xi_{\mathrm{th}}) = 1-(1-\mathrm{ccdf}_{\mathsf{SLM}}(\xi_{\mathrm{th}}))^{N_{\mathrm{T}}} \stackrel{\text{Gauss}}{=} 1-(1-(1-(1-\mathrm{e}^{-\xi_{\mathrm{th}}})^D)^U)^{N_{\mathrm{T}}} \approx N_{\mathrm{T}} D^U \cdot \mathrm{e}^{-\xi_{\mathrm{th}} U} \,. \qquad (4)$$

The above ccdfs (3) and (4) exhibit the same asymptotic slope, determined by the factor U in the exponent ($\log(\mathrm{ccdf}(\xi_{\mathrm{th}})) \approx \mathrm{const.} - \xi_{\mathrm{th}} U$). Compared to SISO SLM, the ccdf of oSLM is worse by the factor N_{T} of antennas.

C. *Directed SLM for MIMO OFDM*

Main idea of *directed SLM (dSLM)* [4] is to invest complexity only where PAR reduction is really needed. Instead of performing U trials for each of the N_{T} transmitters, the fixed budget of $N_{\mathrm{T}}U$ IDFTs (same as in oSLM) is used to successively improve the currently highest PAR over the antennas. Complexity is hence adaptively distributed over the antennas.

For that, in the first step the PARs of the N_T initial (original) OFDM frames are calculated, i.e., ($\mathcal{M}^{(1)}(\boldsymbol{A}) = \boldsymbol{A}$). Then, in each successive step, the OFDM frame with instantaneously highest PAR is considered. Using a next mapping $\mathcal{M}^{(u)}$, a reduction of PAR is tried. This procedure is continued $N_\mathrm{T}(U-1)$ times, leading to the same complexity (in terms of IDFT and PAR calculations) as oSLM. The receiver for dSLM is the same (inverting the applied mapping) as in conventional SLM. For details see [4], [9].

D. Directed SLM for SISO OFDM

Directed SLM can also be applied to SISO systems; we call it *SISO dSLM*. Instead of considering frames over the spatial dimension, in SISO transmission it is possible to group N_t temporally consecutive frames, i.e., to establish *hyper frames*. The straightforward approach is to constitute non-overlapping hyper frames; after each run of dSLM the processing window is shifted by N_t OFDM frames. The same complexity as in original SLM is required ($N_\mathrm{t}U$ candidates per hyper frame of length N_t) but an extra delay is introduced. Here, as the selected OFDM frames are transmitted consecutively and no masking takes place, the average PAR is a sensible criterion. For details on SISO dSLM and for a variant with overlapping hyper frames see [9].

E. CCDF of Directed SLM

To obtain the ccdf of (MIMO or SISO) dSLM, the selection process has to be studied. Given the initial N (equal to N_T or N_t for MIMO or SISO, respectively) OFDM frames, NU candidates are generated from which the best N are selected. We assume the candidates to be pairwise independent, leading to pairwise independent random PAR values ξ_ν. For convenience, we sort the candidates according to increasing PAR (s_λ: index of the λ^th best OFDM frame), i.e., $\xi_{s_1} \leq \xi_{s_2} \leq \ldots \leq \xi_{s_{NU}}$. The ccdf of the PAR of the λ^th best OFDM frame is then given by ($\xi_0 \stackrel{\mathrm{def}}{=} -\infty$)

$$\mathrm{ccdf}_{\mathsf{dSLM},\lambda}(\xi_\mathrm{th}) = \Pr\{\xi_{s_\lambda} > \xi_\mathrm{th}\} = \sum_{l=0}^{\lambda-1} \Pr\{\xi_{s_l} < \xi_\mathrm{th} < \xi_{s_{l+1}}\} \,. \tag{5}$$

The probability for the PAR to lie in a certain interval is given by a binomial distribution with single event probability $1 - \mathrm{ccdf}_{\mathsf{SISO}}(\xi_\mathrm{th})$ and l events out of NU come true. Hence we arrive at

$$\mathrm{ccdf}_{\mathsf{dSLM},\lambda}(\xi_\mathrm{th}) = \sum_{l=0}^{\lambda-1} \binom{NU}{l} \left(1 - \mathrm{ccdf}_{\mathsf{SISO}}(\xi_\mathrm{th})\right)^l \mathrm{ccdf}_{\mathsf{SISO}}^{NU-l}(\xi_\mathrm{th}) \,. \tag{6}$$

1) MIMO dSLM: As already stated, in MIMO OFDM the worst-case PAR is studied. Hence, the ccdf of MIMO dSLM is given by (6) for $\lambda = N = N_\mathrm{T}$, and it reads

$$\mathrm{ccdf}_{\mathsf{MIMO-dSLM}}(\xi_\mathrm{th}) = \mathrm{ccdf}_{\mathsf{SISO}}^{N_\mathrm{T}U}(\xi_\mathrm{th}) \sum_{l=0}^{N_\mathrm{T}-1} \binom{N_\mathrm{T}U}{l} \left(\frac{1-\mathrm{ccdf}_{\mathsf{SISO}}(\xi_\mathrm{th})}{\mathrm{ccdf}_{\mathsf{SISO}}(\xi_\mathrm{th})}\right)^l . \quad (7)$$

2) SISO dSLM: In case of SISO OFDM using dSLM all N_t best candidates are transmitted successively and no worst-case frame masks the others. Not the ccdf of the worst-case PAR is of interest here, but the ccdf over all transmitted candidates. Given the individual statistics of the λ^th best OFDM frame, $\mathrm{ccdf}_{\mathsf{dSLM},\lambda}(\xi_\mathrm{th})$ acc. to (6) for $\lambda = 1, \ldots, N$, the statistics over all frames is given by the respective arithmetic mean, and we have

$$\begin{aligned} \mathrm{ccdf}_{\mathsf{SISO-dSLM}}(\xi_\mathrm{th}) &= \frac{1}{N_\mathrm{t}} \sum_{\lambda=1}^{N_\mathrm{t}} \mathrm{ccdf}_{\mathsf{dSLM},\lambda}(\xi_\mathrm{th}) \quad (8) \\ &= \frac{\mathrm{ccdf}_{\mathsf{SISO}}^{N_\mathrm{t}U}(\xi_\mathrm{th})}{N_\mathrm{t}} \sum_{l=0}^{N_\mathrm{t}-1} (N_\mathrm{t}-l) \binom{N_\mathrm{t}U}{l} \left(\frac{1-\mathrm{ccdf}_{\mathsf{SISO}}(\xi_\mathrm{th})}{\mathrm{ccdf}_{\mathsf{SISO}}(\xi_\mathrm{th})}\right)^l . \end{aligned}$$

3) Asymptotic Behavior of the CCDF: With the approximation used in (3), we have $\mathrm{ccdf}_{\mathsf{SISO}}(\xi_\mathrm{th}) \approx D\mathrm{e}^{-\xi_\mathrm{th}}$. For large values of ξ_th, $1 - \mathrm{ccdf}_{\mathsf{SISO}}(\xi_\mathrm{th})$ tends to one and the sums in either (7) and (8) are well approximated by their last term. Hence, the asymptotic behavior (large ξ_th) of the ccdf is given by

$$\mathrm{ccdf}_{\mathsf{dSLM}}(\xi_\mathrm{th}) \approx C \cdot \binom{NU}{N-1} (D\mathrm{e}^{-\xi_\mathrm{th}})^{NU-(N-1)} = \mathrm{const.} \cdot \mathrm{e}^{-\xi_\mathrm{th}(NU-(N-1))} , \quad (9)$$

with $C = 1$ and $N = N_\mathrm{T}$ for MIMO dSLM and $C = \frac{1}{N_\mathrm{t}}$ and $N = N_\mathrm{t}$ in case of SISO dSLM, respectively. The slope (factor in the exponent) of the ccdf is thus given by $NU - (N-1) = N(U-1) + 1$, which precisely is the maximum number of possible candidates in dSLM if all trials are concentrated on a particular antenna/time slot.

Compared to the slope U of conventional (SISO) SLM or ordinary MIMO SLM, cf. (3), (4), a significant increase to almost NU is obtained, corresponding to faster decay of the ccdf over ξ_th and thus better performance in PAR reduction. Considering the slopes, an effect similar to the diversity order (slope of error rate curves) in MIMO communication is present in MIMO/hyper frame PAR reduction as well. However, similar as for error rates (power loss or SNR gap), the ccdf of dSLM is not simply that for SISO SLM using $N(U-1)+1$ candidates, but, due to the factor $C \cdot \binom{NU}{N-1} D^{NU-(N-1)}$ in (9), it additionally exhibits a horizontal shift.

IV. Numerical Results

The performance of the proposed versions of SLM is now assessed by means of numerical simulations. The number of carriers (all active) is $D = 512$ and the modulation in each carrier is 4PSK. W.l.o.g. SLM applying phase vectors is studied,[3] where the phase vectors are chosen randomly with elements drawn from the set $\{\pm 1,\ \pm \mathrm{j}\}$ (pure inversion or interchange of the quadrature components [2]).

A. MIMO Transmission

For MIMO transmission, $N_\mathrm{T} = 4$ parallel data streams are assumed. Choosing $U = 4$ or 16 phase vectors, in each PAR reduction scheme $4 \cdot 4 = 16$, and $4 \cdot 16 = 64$ IDFTs, respectively, have to be calculated per OFDM frame duration. In Fig. 1 the ccdfs of the worst-case PAR for ordinary and directed SLM are displayed. It is clearly visible that for the same number of IDFTs (complexity) dSLM shows (much) better performance than oSLM. For clipping levels lower than 10^{-5} and $U = 4$ gains of more than 1 dB over oSLM are achievable. The larger slope of the dSLM ccdf is also noticeable. The simulations fit well with the respective above given analytic results (gray solid) and the approximations (asymptotes) of the theoretical curves (gray dashed).

To illustrate the dependence on the number of antennas, in Fig. 2 the ccdf of PAR is plotted for $N_\mathrm{T} = 2$, 4, and 8. The number of candidates is chosen to $U = 4$. As derived above, increasing N_T leads to higher worst-case PARs in conventional MIMO OFDM and oSLM, cf. (4). Conversely, using dSLM, performance improves for larger numbers of antennas and the steepness of the curves increases to almost $N_\mathrm{T} U$.

B. SISO Transmission

In Fig. 3, the ccdf of SISO dSLM is shown for different numbers U of candidates and hyper frame lengths N_t ($N_\mathrm{t} = 1$ corresponds to conventional SISO SLM). The performance of (d)SLM increases either if more candidates are taken into account (larger U, higher computational complexity) or if a larger hyper frame length N_t is used. In this case complexity remains constant but a delay of $N_\mathrm{t} - 1$ OFDM frames is introduced. Directed SLM provides

[3] The performance results are representative for all types of mappings $\mathcal{M}$ (phase vectors, permutations, scramblers, etc.), as long as the generated candidates are (almost) independent. Cf. also [4, Fig. 8].

a significant gain compared to original SLM, about $1\,\mathrm{dB}$ for $U = 8$ and $N_\mathrm{t} = 8$ at a clipping levels $\Pr\{\xi > \xi_\mathrm{th}\} = 10^{-5}$.

In order to assess which parameter combination (U, N_t) is preferably used in PAR reduction, Fig. 4 shows a contour plot of the threshold ξ_th at a clipping probability of $\Pr\{\xi > \xi_\mathrm{th}\} = 10^{-4}$ over U and N_t. These results are based on the analytic expression (8). Using the original SLM (i.e., $N_\mathrm{t} = 1$) with large values of U, it is difficult to obtain additional gains by further increasing U (going to the right on the horizontal axis). This can be circumvented by applying dSLM (going in vertical direction in the contour plot). It is hence reasonable to enlarge the number U of candidates only together with an appropriate choice of N_t. In addition, the trajectory of steepest descent starting from conventional OFDM ($U = 1$, $N_\mathrm{t} = 1$), is depicted. It gives a hint on which pairs $(U,\ N_\mathrm{t})$ should preferably be used. Since it is almost a straight line, it is sensible to choose N_t proportional to U.

V. Conclusions

In this paper, the performance of SLM operating jointly on blocks of OFDM frames, either in spatial or temporal direction, has been assessed. Analytic expressions for the ccdf of PAR have been derived. In contrast to other MIMO PAR reduction schemes, using directed SLM the ccdf of PAR exhibits a steeper decay, increased by a factor (almost) equal to the number of transmit antennas, i.e., “diversity gain” is achieved. In SISO transmission, a trade-off between complexity and delay is additionally possible.

References

[1] M.-S. Baek, M.-J. Kim, Y.-H. You, H.-K. Song. Semi-Blind Channel Estimation and PAR Reduction for MIMO-OFDM System With Multiple Antennas. *IEEE Transactions on Broadcasting*, pp. 414–424, Dec. 2004.

[2] R. Bäuml, R.F.H. Fischer, J.B. Huber. Reducing the Peak-to-Average Power Ratio of Multicarrier Modulation by Selected Mapping. *IEE Electronics Letters*, pp. 2056–2057, Nov. 1996.

[3] J.A.C. Bingham. Multicarrier Modulation for Data Transmission: An Idea Whose Time Has Come. *IEEE Communications Magazine*, pp. 5–14, May 1990.

[4] R.F.H. Fischer, M. Hoch, Peak-to-Average Power Ratio Reduction in MIMO OFDM. In *IEEE International Conference on Communications (ICC 2007)*, Glasgow, United Kingdom, June 2007.

[5] S.H. Han, J.H. Lee, An Overview of Peak-to-Average Power Ratio Reduction Techniques for Multicarrier Transmission. *IEEE Wireless Communications*, pp. 56–65, April 2005.

[6] Y.-L. Lee, Y.-H. You, W.-G. Jeon, J.-H. Paik, H.-K. Song. Peak-to-Average Power Ratio in MIMO-OFDM Systems Using Selective Mapping. *IEEE Communications Letters*, pp. 575–577, Dec. 2003.

[7] S.H. Müller, R.W. Bäuml, R.F.H. Fischer, J.B. Huber. OFDM with Reduced Peak-to-Average Power Ratio by Multiple Signal Representation. *Annal of Telecommunications*, pp. 58–67, Feb. 1997.

[8] H. Ochiai. Performance Analysis of Peak Power and Band-Limited OFDM Systems with Linear Scaling. *IEEE Transactions on Communications*, pp. 1055–1065, Sep. 2003.

[9] C. Siegl, R.F.H. Fischer. Application of Directed Selected Mapping for Peak-to-Average Power Ratio Reduction in Single Antenna OFDM. In *12th International OFDM Workshop*, Hamburg, Germany, Aug. 2007.

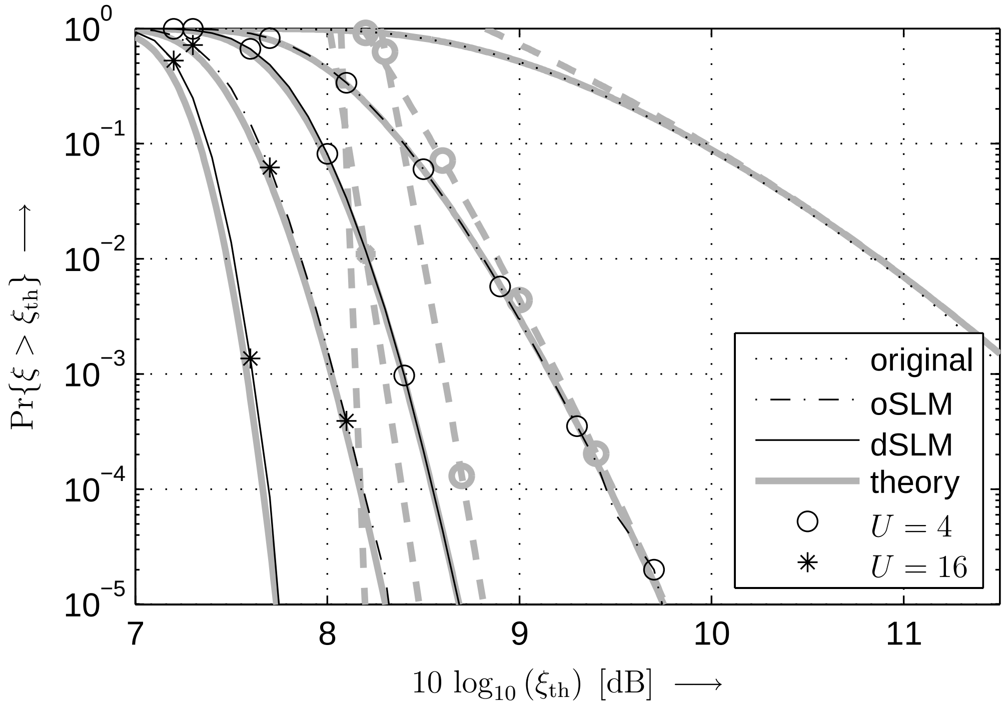

Fig. 1. Ccdf of PAR for MIMO OFDM with dSLM and oSLM. $D = 512$ carriers, $N_{\mathrm{T}} = 4$ antennas, $U = 4$, 16 candidate vectors.

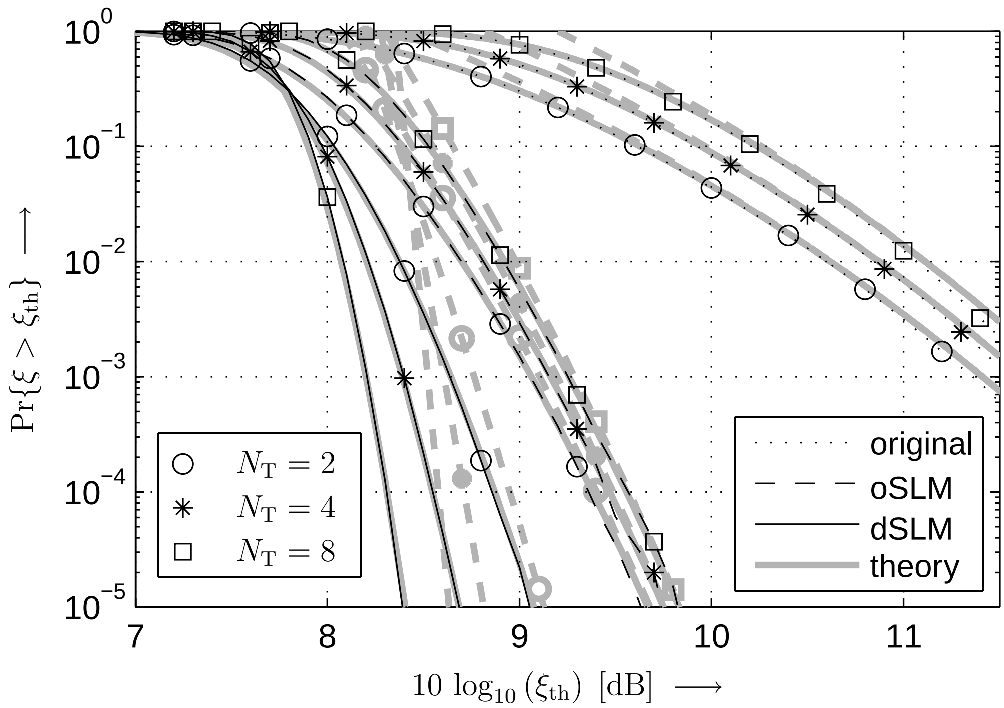

Fig. 2. Ccdf of PAR for MIMO OFDM with dSLM and oSLM. $D = 512$ carriers, $N_{\mathrm{T}} = 2$, 4, and 8 antennas, $U = 4$ candidate vectors.

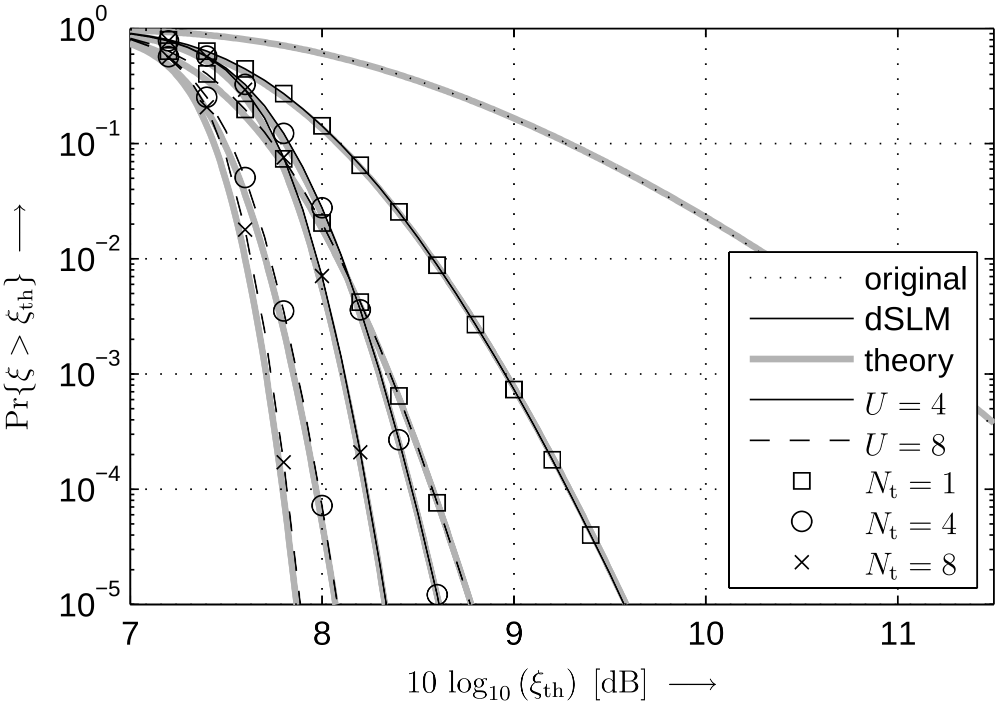

Fig. 3. Ccdf of PAR of original SLM ($N_{\rm t} = 1$) and SISO dSLM. $D = 512$ and 4PSK.

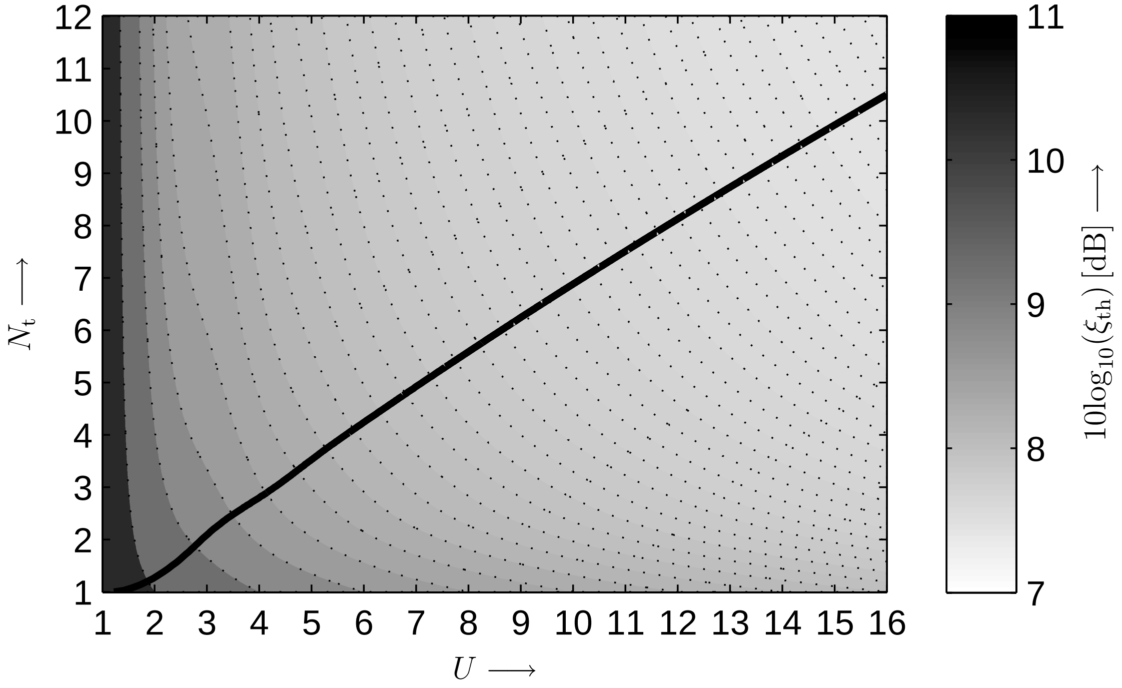

Fig. 4. Contour plot of ξ_th at clipping probability $\Pr\{\xi > \xi_\mathrm{th}\} = 10^{-4}$ over U and N_t derived from the theoretical expression (8). The trajectory of steepest descent is depicted, too.