Bit Error Probability of a Rapidly-Adapting Rake Receiver as a Function of Dwell Time

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Abstract. Implementable rake receivers have less correlators than there are propagation paths in the channel. In the case of slowly time-variant channels the rake receiver is able to adapt to the channel by selecting the paths with the maximum momentary power [Gue92a, Gue92b]. In practice, it is not possible to select the best paths for every modulation interval. The bit error probability of rake receivers is calculated for the situation that the path selection is maintained over fixed time periods, the dwell time. The results show that the degradation to an instantanous adaption is insignificant as long as the dwell time is less than five per cent of the inverse of the fading bandwidth.

1 Introduction

The spread spectrum communication technique in combination with rake receivers is an efficient method to reduce the bit error rate of digital transmission schemes over noisy fading channels by exploiting the inherent diversity of multipath propagation. Therefore, code division multiple access (CDMA) with nearly orthogonal spreading carriers for the different users is a candidate for future mobile communication systems.

The number of correlators of an implementable rake receiver will usually be small compared to the number of propagation paths of the channel. Thus, a rake receiver has to select the paths which have the largest received power and to perform maximum-ratio combining of these selected paths. On channels with fast fading it is only possible to select the paths with largest average power. In the following such a receiver is called a *classical rake*. In [Gue92a, Gue92b] a new structure for slowly time-variant channels is introduced where the paths with the maximum momentary power are selected. This is called the *rapidly-adapting rake receiver (RA-rake)*.

In [Gue92a, Gue92b] the performance of the RA-rake is studied under the assumption that in every modulation interval the paths with largest power are selected. In a practical application this is obviously not possible because there are only few correlators to scan all propagation paths. The purpose of this paper is to study the influence of time for scanning all paths. Because within this time the correlators dwell on the — perhaps meanwhile suboptimum — paths, this period is called the dwell time of a RA-rake.

In [BP89] a similar work is given for schemes with selection diversity which may be interpreted as a simplified case of the results presented in this paper.

2 Assumptions

Coherent 2PSK modulation over a slowly time-variant frequency-selective Rayleigh fading channel is assumed. The L independently fading propagation paths are complex-valued Gaussian processes with power density spectra given in [SBS66]. The corresponding autocorrelation function (acf) $\varphi(\Delta t)$ of each quadrature component of each of the L propagation paths is given by

$$\varphi(\Delta t) = \frac{1}{2L} \exp\left\{ -\frac{-\pi^2 B^2 \Delta t^2}{\ln 2} \right\} \tag{1}$$

where B is the 3 dB fading bandwidth. In this model, it is assumed that all L paths contribute the same average power. Therefore, the multipath propagation corresponds to L-diversity.

To select the L_s paths with maximum momentary power it is necessary to scan all L possible propagation paths. In practice, there will be used more or less complex scanning strategies. Here we assume a simplified model. The receiver selects every T_d seconds the L_s paths with maximum momentary power. In the time between, the selected paths are maintained and the receiver performs maximum-ratio combining of these paths. Thus, in the following T_d is called the dwell time. Relating to the maximum-ratio combining we assume that the channel weights $g_\lambda(t)$ of the selected paths are ideally estimated; $\lambda \in \{1, \ldots, L\}$ is the index of the propagation paths of the channel.

3 Time-dependent probability density function

The maximum-ratio combining with periodic selection of the paths is characterized by the utilizable power of the receiver input signal

$$\eta(t) = \sum_{\mu=1}^{L_s} \left| g_{\lambda_\mu}(t) \right|^2 \tag{2}$$

where at the instants $t = \kappa T_d$, $\kappa \in \mathbb{Z}$ the L_s paths with the indices $\lambda_1, \ldots, \lambda_{L_s}$ are selected according to the criterion of maximum utilizable power $\eta(\kappa T_d)$:

$$\eta(\kappa T_d) = \sum_{\mu=1}^{L_s} \left| g_{\lambda_\mu}(\kappa T_d) \right|^2 = \max_{\substack{\lambda_1, \ldots, \lambda_{L_s} \in \{1, \ldots, L\} \\ \lambda_\mu \neq \lambda_\nu \, \forall \, \mu \neq \nu}} \sum_{\mu=1}^{L_s} \left| g_{\lambda_\mu}(\kappa T_d) \right|^2 \tag{3}$$

The statistical parameters of the random process $\eta(t)$ change with t but depend only on the difference of the actual time to the moment of the last path selection.

Therefore, $\eta(t)$ is a cyclostationary process and we introduce $t = \kappa T_d + \Delta t$ where $\Delta t \in [0; T_d)$. Without loss of generality, we only consider the case $\kappa = 0$.

The probability density function (pdf) of the quadrature components $\mathrm{Re}\{g_\lambda(\Delta t)\}$ and $\mathrm{Im}\{g_\lambda(\Delta t)\}$ of each propagation path for given values $g_\lambda(0)$ is Gaussian with mean $\rho(\Delta t)\mathrm{Re}\{g_\lambda(0)\}$ or $\rho(\Delta t)\mathrm{Im}\{g_\lambda(0)\}$ and variance $\sigma^2(\Delta t)$ with [Pap91]

$$\begin{aligned} \rho(\Delta t) &= \varphi(\Delta t)/\varphi(0) \\ \sigma^2(\Delta t) &= \varphi(0) - \varphi^2(\Delta t)/\varphi(0). \end{aligned}$$

From this it follows that the pdf of $\eta(\Delta t)$, which is the sum of $2L_s$ squared quadrature components with the same pdf (see (2)), for given $g_{\lambda_\mu}(0)$ is a noncentral χ^2-distribution with the noncentrality parameter $\eta(0)$ [Pro89]. Thus, this pdf is only dependent on the random variable $\eta(0)$:

$$\begin{aligned} p_{\eta(\Delta t)|\eta(0)}(u|v) \;=\; & \frac{1}{2\sigma^2(\Delta t)} \left[\frac{u}{\rho^2(\Delta t) v}\right]^{\frac{L_s-1}{2}} \exp\left\{-\frac{u+\rho^2(\Delta t)v}{2\sigma^2(\Delta t)}\right\} \\ & I_{L_s-1}\left(\rho(\Delta t)\frac{\sqrt{uv}}{\sigma^2(\Delta t)}\right); \qquad u \geq 0,\, v \geq 0 \end{aligned} \tag{4}$$

Here, $I_n(\cdot)$ is the modified bessel function of first kind and order n.

The pdf of $\eta(\Delta t)$ is calculated by

$$p_{\eta(\Delta t)}(u) = \int_0^\infty p_{\eta(\Delta t)|\eta(0)}(u|v) p_{\eta(0)}(v)\,\mathrm{d}v \tag{5}$$

where $\eta(0)$ is determined by selecting and combining the instantanous best paths. Its pdf is given in [Gue92a, Gue92b].

This leads to

Theorem 1: *For any $u \geq 0$ and $L_s < L$ the pdf of the utilizible power of the receiver input signal of a RA-rake receiver is*

$$\begin{aligned} p_{\eta(\Delta t)}(u) = & \frac{L!\,L}{L_s!\,L_s} \frac{1}{[2\sigma^2(\Delta t)]^{L_s}} \sum_{\mu=0}^{\infty} u^{L_s+\mu-1} \exp\left\{-\frac{u}{2\sigma^2(\Delta t)}\right\} \sum_{\kappa=1}^{L-L_s} a_\kappa \\ & \left\{ \left(-\frac{L_s}{\kappa}\right)^{L_s} \frac{1}{(L_s+\mu-1)!} \frac{\left(\frac{\rho^2(\Delta t)}{4\sigma^4(\Delta t)}\right)^\mu}{\left[(\frac{\kappa}{L_s}+1)L + \frac{\rho^2(\Delta t)}{2\sigma^2(\Delta t)}\right]^{\mu+1}} \right. \\ & \left. - \sum_{\nu=1}^{L_s} \left(-\frac{L_s}{\kappa}\right)^\nu \frac{L^{L_s-\nu}}{(L_s-\nu)!} \frac{(L_s+\mu-\nu)!}{(L_s+\mu-1)!\,\mu!} \frac{\left(\frac{\rho^2(\Delta t)}{4\sigma^4(\Delta t)}\right)^\mu}{\left[L + \frac{\rho^2(\Delta t)}{2\sigma^2(\Delta t)}\right]^{L_s-\nu+\mu+1}} \right\} \end{aligned} \tag{6}$$

with

$$a_\kappa = \frac{(-1)^{\kappa-1}}{(\kappa-1)!\,(L-L_s-\kappa)!}. \tag{7}$$

The proof is given in appendix A.

4 Bit Error Probability

For deterministic channel weights g_λ the probability of error is given by [Pro89]

$$P_b = Q\left(\sqrt{2\frac{E_b}{N_0}\eta}\right) \tag{8}$$

where $Q(\cdot)$ is the Gaussian integral function

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \mathrm{e}^{-w^2/2}\mathrm{d}w. \tag{9}$$

In our case $\eta = \eta(\Delta t)$ is a random process and thus we have to calculate the expected value of the bit error probability at time Δt by

$$P_b(\Delta t) = \int_0^\infty Q\left(\sqrt{2\frac{E_b}{N_0}u}\right) p_{\eta(\Delta t)}(u)\,\mathrm{d}u. \tag{10}$$

Straight forward calculations shown in appendix B lead to

Theorem 2: *The probability of error for the RA-rake at time Δt after the selection of the L_s best of L paths in case $L_s < L$ is given by*

$$\begin{aligned} P_b(\Delta t) \;=\; & \frac{1}{2}\frac{L!\,L}{L_s!\,L_s}\sum_{\mu=0}^{\infty}\left(1-\sqrt{\frac{2\sigma^2(\Delta t)E_b/N_0}{2\sigma^2(\Delta t)E_b/N_0+1}}\sum_{\lambda=0}^{L_s+\mu-1}\frac{(2\lambda)!}{2^{2\lambda}(\lambda!)^2}\right. \\ & \left.\left(\frac{1}{1+2\sigma^2(\Delta t)E_b/N_0}\right)^{\lambda}\right)\sum_{\kappa=1}^{L-L_s} a_\kappa\left\{-\left(\frac{L_s}{\kappa}\right)^{L_s}\right. \\ & \frac{\left(\frac{\rho^2(\Delta t)}{2\sigma^2(\Delta t)}\right)^{\mu}}{\left[\left(\frac{\kappa}{L_s}+1\right)L+\frac{\rho^2(\Delta t)}{2\sigma^2(\Delta t)}\right]^{\mu+1}} - \sum_{\nu=1}^{L_s}\left(-\frac{L_s}{\kappa}\right)^{\nu}\frac{L^{L_s-\nu}}{(L_s-\nu)!} \\ & \left.\frac{(L_s+\mu-\nu)!}{\mu!}\frac{\left(\frac{\rho^2(\Delta t)}{2\sigma^2(\Delta t)}\right)^{\mu}}{\left[L+\frac{\rho^2(\Delta t)}{2\sigma^2(\Delta t)}\right]^{L_s-\nu+\mu+1}}\right\}. \end{aligned} \tag{11}$$

In case $L_s \geq L$, the RA-rake performs like the classical rake and the bit error probability becomes independent of Δt.

From Theorem 2, the average bit error probability

$$\bar{P}_b = \frac{1}{T_d}\int_0^{T_d} P_b(\Delta t)\,\mathrm{d}\Delta t. \tag{12}$$

can be calculated by standard numerical methods.

It should be remarked that in case T_d becomes zero the average bit error probability tends to the result in [Gue92a, Gue92b] for path selection in each modulation interval whereas in case that T_d goes to infinity the RA-rake receiver performs like the classical rake.

5 Results

The average bit error probability of a RA-rake receiver has been determined for various parameters. The figures 1 to 3 show the resulting gain by rapidly adapting to the propagation paths with maximum momentary power over the classical rake, called RA-gain, versus the dwell time. The dwell time T_d is normalized on the fading bandwidth B. In all cases, the receiver achieves for $T_d B \leq 0.01$ the same performance as an ideal RA-rake. For $T_d B > 0.1$ the RA-gain decreases and the RA-rake receiver tends towards the classical rake.

Figure 1 shows the RA-gain for various bit error probabilities for fixed numbers of correlators and propagation paths. For low bit error probability the RA-gain decreases at smaller dwell times than in case of high bit error probability.

In the figures 2 and 3 the RA-gain is plotted for fixed bit error probability of 10^{-4} and various choices of the numbers of correlators and propagation paths. Although the maximum RA-gains for the various parameters are very different, the loss by increasing dwell time is similar. Thus, we can give the rule of thumb that to limit the degradation by a finite dwell time to less than 0.2 dB (at a bit error probability of 10^{-4}) the dwell time has to be chosen to $T_d \leq 0.05/B$. (The fading bandwidth B is a function of the carrier frequency and the mobile velocity.)

Furthermore, we remark that the performance of the receiver directly depends on the acf $\varphi(\Delta t)$. On the other hand, in case $\Delta t \leq 0.3/B$ the acf depends approximately only on the fading bandwidth and not on the exact form of the Doppler spectrum. Because we are interested only in the values of the dwell time for which the performance of the receiver decreases first, it is possible to generalize the presented results for the Doppler spectrum of [SBS66] to other typical Doppler spectra.

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Appendix

A Proof of Theorem 1

To derive the pdf of the utilizable power $\eta(\Delta t)$ defined in eqn. (2) we start by inserting the arguments of the integral in eqn. (5). The first argument is given in eqn. (4) and the second one is derived as theorem 1 in [Gue92a]:

$$p_{\eta(0)}(v) = \frac{L!\,L}{L_s!\,L_s} \sum_{\kappa=1}^{L-L_s} a_\kappa \left\{ \left(-\frac{L_s}{\kappa}\right)^{L_s} \exp\left\{-\left(\frac{\kappa}{L_s}+1\right) Lv\right\} - \sum_{\nu=1}^{L_s} \left(-\frac{L_s}{\kappa}\right)^{\nu} \frac{1}{(L_s-\nu)!} (Lv)^{L_s-v} \exp\{-Lv\} \right\} \quad (13)$$

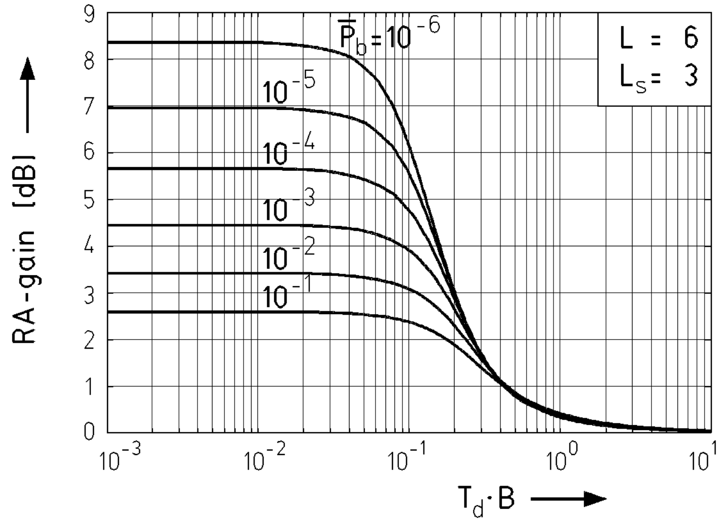

Fig. 1. Gain by rapidly adapting to the propagation paths with maximum momentary power over the classical rake (RA-gain) versus normalized dwell time of the adaption process for various bit error probabilities if the RA-rake has three correlators and there are six propagation paths

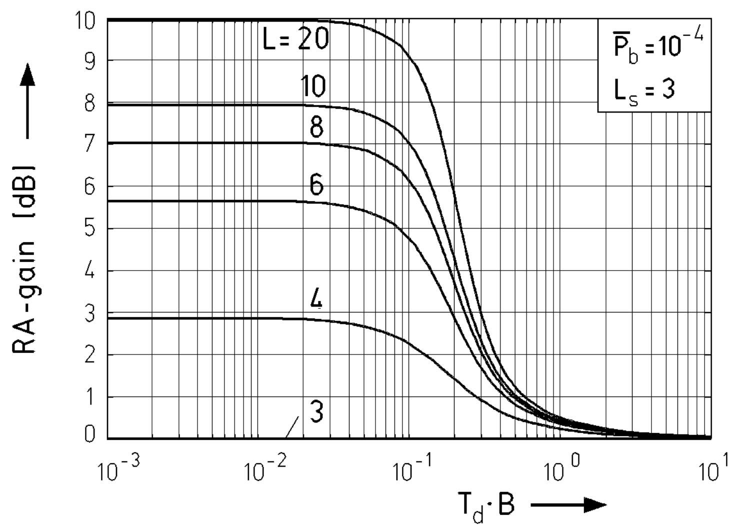

Fig. 2. RA-gain versus normalized dwell time for various numbers of propagation paths at the bit error probability of 10^{-4} if the RA-rake has three correlators

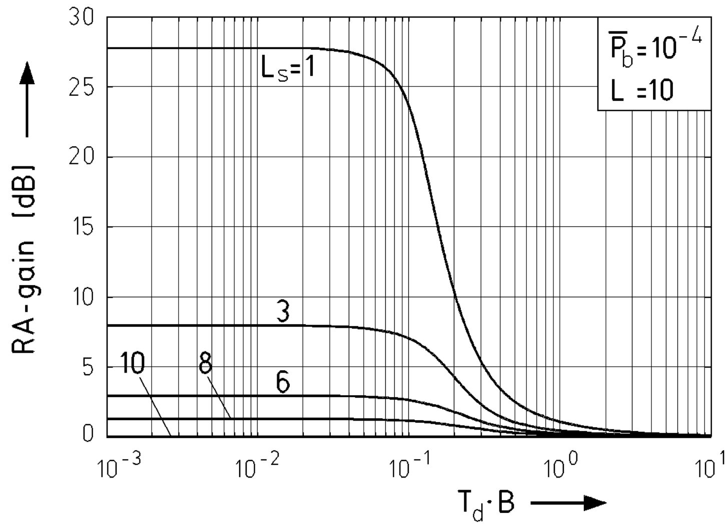

Fig. 3. RA-gain versus normalized dwell time for various numbers of correlators at the bit error probability of 10^{-4} in case of ten propagation paths

where a_κ is given in eqn. (7).
After few rearrangements it follows

$$
\begin{aligned}
p_{\eta(\Delta t)}(u) = {} & \frac{L!\,L}{L_s!\,L_s}\frac{\rho^{1-L_s}}{2\sigma^2}u^{\frac{L_s-1}{2}}\exp\left\{-\frac{u}{2\sigma^2}\right\}\sum_{\kappa=1}^{L-L_s} a_\kappa \Bigg\{ \left(-\frac{L_s}{\kappa}\right)^{L_s} \\
& \int_0^\infty v^{-\frac{L_s-1}{2}} \exp\left\{-\left[\left(\frac{\kappa}{L_s}+1\right)L+\frac{\rho^2}{2\sigma^2}\right]v\right\} I_{L_s-1}\left(\frac{\rho\sqrt{u}}{\sigma^2}\sqrt{v}\right)\,\mathrm{d}v \\
& -\sum_{\nu=1}^{L_s}\left(-\frac{L_s}{\kappa}\right)^{\nu}\frac{L^{L_s-\nu}}{(L_s-\nu)!}\int_0^\infty v^{-\nu+\frac{L_s+1}{2}}\exp\left\{-\left[L+\frac{\rho^2}{2\sigma^2}\right]v\right\} \\
& \qquad I_{L_s-1}\left(\frac{\rho\sqrt{u}}{\sigma^2}\sqrt{v}\right)\,\mathrm{d}v \Bigg\}
\end{aligned}
\tag{14}
$$

where for simplified and shorter expressions the dependency of Δt in $\rho(\Delta t)$ and $\sigma(\Delta t)$ is omitted. Now, the integration variable is changed by substituting $w = \sqrt{v}$.

$$
\begin{aligned}
p_{\eta(\Delta t)}(u) = {} & \frac{L!\,L}{L_s!\,L_s}\frac{\rho^{1-L_s}}{\sigma^2}u^{\frac{L_s-1}{2}}\exp\left\{-\frac{u}{2\sigma^2}\right\}\sum_{\kappa=1}^{L-L_s} a_\kappa \Bigg\{ \left(-\frac{L_s}{\kappa}\right)^{L_s} \\
& \int\limits_0^\infty w^{-L_s+2}\exp\left\{-\left[\left(\frac{\kappa}{L_s}+1\right)L+\frac{\rho^2}{2\sigma^2}\right]w^2\right\} I_{L_s-1}\left(\frac{\rho\sqrt{u}}{\sigma^2}w\right)\,\mathrm{d}w \\
& -\sum_{\nu=1}^{L_s}\left(-\frac{L_s}{\kappa}\right)^{\nu}\frac{L^{L_s-\nu}}{(L_s-\nu)!}\int\limits_0^\infty w^{L_s-2\nu+2}\exp\left\{-\left[L+\frac{\rho^2}{2\sigma^2}\right]w^2\right\} \\
& \qquad I_{L_s-1}\left(\frac{\rho\sqrt{u}}{\sigma^2}w\right)\,\mathrm{d}w\Bigg\}
\end{aligned} \tag{15}
$$

Using the series expansion of the modified bessel function in [Pro89], p. 28

$$
I_n(x) = \sum_{\mu=0}^{\infty}\frac{(x/2)^{n+2\mu}}{\mu!\,\Gamma(n+\mu+1)} \qquad \text{for} \qquad x \geq 0 \tag{16}
$$

where $\Gamma(\cdot)$ is the Gamma function, we can evaluate the integral

$$
\begin{aligned}
\int\limits_0^\infty w^{m-1}\mathrm{e}^{-\alpha w^2} I_n(\beta w)\,\mathrm{d}w \; = \; & \frac{\beta^n}{2^n}\sum_{\mu=0}^{\infty}\frac{\beta^{2\mu}}{\mu!\,\Gamma(n+\mu+1)\,2^{2\mu}} \\
& \int\limits_0^\infty w^{m+n+2\mu-1}\mathrm{e}^{-\alpha w^2}\,\mathrm{d}w.
\end{aligned} \tag{17}
$$

The integral on the right-hand side of eqn. (17) is given in [BS85], p. 66

$$
\int_0^\infty w^n \mathrm{e}^{-\alpha w^2}\,\mathrm{d}w = \frac{\Gamma(\frac{n+1}{2})}{2\,\alpha^{\frac{n+1}{2}}} \quad \text{for} \quad \alpha > 0;\; n > -1. \tag{18}
$$

Putting this equation in (17) leads to the interim result

$$
\int\limits_0^\infty w^{m-1}\mathrm{e}^{-\alpha w^2} I_n(\beta w)\,\mathrm{d}w = \frac{\beta^n}{2^{n+1}\,\alpha^{\frac{m+n}{2}}}\sum_{\mu=0}^{\infty}\frac{\Gamma\left(\frac{m+n}{2}+\mu\right)}{\Gamma\left(n+\mu+1\right)}\frac{\left(\frac{\beta^2}{4\alpha}\right)^{\mu}}{\mu!} \tag{19}
$$

Using eqn. (19) and the fact that for integer arguments the Gamma function is given by $\Gamma(n+1) = n!$, from eqn. (15) follows theorem 1 and the proof is completed.

B Proof of Theorem 2

To calculate the bit error probability we have to evaluate eqn. (10). After inserting the expression of theorem 1, it is obviously necessary to evaluate the following integral which solution is given in [Gue92a], Lemma 2:

$$\int_0^\infty u^n \mathrm{e}^{-\beta u} Q\left(\sqrt{2\frac{E_b}{N_0}u}\right)\,\mathrm{d}u = \frac{1}{2}\frac{n!}{\beta^{n+1}}\left(1-\sqrt{\frac{E_b/N_0}{E_b/N_0+\beta}}\sum_{\lambda=0}^{n}\frac{(2\lambda)!}{2^{2\lambda}(\lambda!)^2}\frac{\beta^\lambda}{(E_b/N_0+\beta)^\lambda}\right) \quad (20)$$

Few rearrangements leads directly to the expression of theorem 2 and completes the proof.

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