Capacity of Clipped 4-QAM-OFDM

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***Abstract*— The capacity of OFDM with 4-QAM modulation per carrier over channels with clipping at the transmitter side is studied. For this, the individual capacities of the OFDM frames are considered. It is shown that capacity is linearly related to the energy of the OFDM frame after clipping. Based on the insight that the power leakage is the main source of the loss, via geometrical considerations a very tight upper bound on the capacity of clipped OFDM is derived. Clipped OFDM performs the same as unclipped OFDM with accordingly reduced signal-to-noise ratio.**

I. Introduction

The performance of orthogonal frequency-division multiplexing (OFDM) in the presence of nonlinear amplifiers at the transmitter is of major interest for future communication systems. Many authors have studied this problem from different perspectives such as error rates, signal-to-noise ratios or capacities, e.g., [6], [4], [13], [14], [15], the optimum receiver design and signal reconstruction at the receiver, e.g., [3], [17], [7], or optimization of clipped systems, e.g., [12], [18]. Receiver impairments, e.g., [10], have also been analyzed.

In most of the literature, a *statistical clipping model* is used, i.e., similar to quantization, the memoryless nonlinear distortion is treated as additional noise. Conversely, like in [14], we deal with clipping as what it actually is, a *deterministic operation*. Thereby much tighter bounds on the (potential) performance of OFDM impaired by transmitter side nonlinearities are obtained.

In this paper, the capacity of OFDM with 4-QAM modulation per carrier over channels with clipping is studied. To this end, the *individual capacities* of the OFDM frames are assessed. It is shown that these capacities are linearly related to the energy of the respective OFDM frame after clipping. This fact can immediately be used, e.g., for signal selection or shaping, i.e., algorithmic signal design in OFDM. Based on the insight that the power leakage is the main source of the loss, a very tight upper bound on the capacity of clipped OFDM is derived. This bound is based on geometrical considerations, in particular the arrangement of sum of Gaussian densities in signal space. The main important result (anticipated in [14] but not proven) is that clipped OFDM performs the same as unclipped OFDM with signal-to-noise ratio reduced according to the clipping power loss.

The paper is organized as follows: In Section II, the system model is introduced. The individual capacities of the OFDM frames are derived and discussed in Section III. Bounds on the capacity of clipped OFDM are derived in Section IV; some conclusions and extensions are given in Section V.

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II. System Model

Figure 1 shows the OFDM system under consideration. Throughout the paper, a standard discrete-time OFDM system model based on an (I)DFT of length D [1], [16] is assumed. All carriers are expected to be active and modulation with 4-QAM, i.e., $A_d \in \mathcal{A} \stackrel{\text{def}}{=} \{\pm 1 \pm \text{j}\}$, $d = 0, \dots, D-1$, is exclusively assumed ($\sigma_a^2 = 2$). The data symbols are combined into the frequency-domain OFDM frame, denoted by $\boldsymbol{A} = [A_0, \dots, A_{D-1}]$, which is transformed into the time-domain OFDM frame $\boldsymbol{a} = [a_0, \dots, a_{D-1}] = \text{IDFT}\{\boldsymbol{A}\}$ via IDFT, i.e., $a_k = \frac{1}{\sqrt{D}} \sum_{d=0}^{D-1} A_d \cdot \text{e}^{\text{j}2\pi kd/D}$, $k = 0, \dots, D-1$.

After parallel-to-serial conversion (the insertion of the guard interval is ignored for the moment), the transmit symbols are affected by a nonlinear, memoryless device $g(\cdot)$, modeling, e.g., the power amplifier. The distorted samples are then given by

$$z_k = g(a_k) = \gamma_k \cdot a_k \;, \tag{1}$$

where $0 \leq \gamma_k \leq 1$ is a real-valued scaling factor (dependent on a_k; only amplitude distortion is assumed here) characterizing the clipping of the actual symbol.

Subsequently we study the performance of OFDM when an ideal soft limiter is present, i.e.,

$$g(a) = \begin{cases} a, & |a| \leq G \\ G \cdot a/|a|, & |a| > G \end{cases} \;, \tag{2}$$

where G is the clipping level. However, the presented results are likewise valid for any other type of nonlinearity exhibiting some saturation.

The distorted transmit symbols are fed into the channel with transfer function $H(z)$, respectively impulse response h_k, and the receive symbols are corrupted by additive white Gaussian noise with zero mean and variance σ_n^2 per complex sample, statistically independent of the transmit signal. In summary we have ($*$ denotes convolution)

$$y_k = z_k * h_k + n_k = (\gamma_k \cdot a_k) * h_k + n_k \tag{3}$$

Via DFT, the received block is transformed into frequency domain, resulting in the frequency-domain receive vector $\boldsymbol{Y}$ with components ($H_d \stackrel{\text{def}}{=} H(\text{e}^{\text{j}2\pi d/D})$ is the frequency-domain scaling of the dth subcarrier; $\boldsymbol{H} = [H_0, \dots, H_{D-1}]$)

$$Y_d = H_d \cdot Z_d + N_d = H_d \cdot \alpha_d A_d + N_d \;. \tag{4}$$

The complex-valued scaling factor α_d in subcarrier d is dependent on the actual transmitted OFDM frame. Assuming a particular OFDM frame $\boldsymbol{A}_i \in \mathcal{A}^D$ has been transmitted, from $Z_{d,i} = \text{DFT}\{z_{k,i}\} = \text{DFT}\{\gamma_{k,i} a_{k,i}\} = \alpha_{d,i} A_{d,i}$, the $\alpha_{d,i}$'s (describing the result of a convolution of $\text{DFT}\{\gamma_{k,i}\}$ with $\boldsymbol{A}_i$) can be calculated. Combining the scaling factors

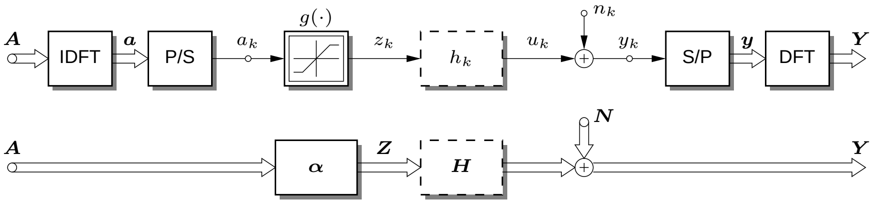

Fig. 1. OFDM system model with nonlinear operation on the transmit symbols. Top: time-domain description; bottom: frequency-domain model.

into a vector $\boldsymbol{\alpha}_i \overset{\text{def}}{=} [\alpha_{0,i}, \ldots, \alpha_{D-1,i}]$, each possible OFDM frame $\boldsymbol{A}_i$ is associated with a vector $\boldsymbol{\alpha}_i$. Noteworthy, given the actual input, clipping is a deterministic operation, hence the factors $\alpha_{d,i}$ are deterministic and can be calculated for each given setting (cf. [14]).

Subsequently, we first assume an AWGN channel, i.e., $H_d = 1$, $\forall d$. We hence study the transmission of each particular OFDM frame $\boldsymbol{A}_i$, $i = 1, \ldots, 4^D$, which is given in frequency domain as

$$Y_{d,i} = Z_{d,i} + N_d = \alpha_{d,i} A_{d,i} + N_d \;. \tag{5}$$

Based on the AWGN results, the extension to linearly distorting channels (fading channels over frequency) is possible.

III. Capacity and Signal Selection

In this section, we study the capacity of clipped OFDM. In particular, we are interested in the contribution of each particular OFDM frame to the average capacity.

A. No Clipping

We first assume an AWGN channel and that no clipping is present. Moreover, we always expect a uniform distribution of the data symbols A_d, independently over the carriers, which leads to a uniform distribution of the OFDM frames, i.e., $\Pr\{\boldsymbol{A}_i\} = 1/4^D$.

From basic information theory [5] we have for the capacity of the OFDM scheme (in bits per complex dimension; $\mathrm{f}.(\cdot)$ denotes probability density function (pdf))

$$C_{\text{OFDM}}(\tfrac{\sigma_a^2}{\sigma_n^2}) = \frac{1}{D}\int\!\cdots\!\int \sum_{i=1}^{4^D} \frac{1}{4^D} \mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{A}_i) \cdot \log_2\left(\frac{\mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{A}_i)}{\sum_{l=1}^{4^D}\frac{1}{4^D}\mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{A}_l)}\right) \mathrm{d}\boldsymbol{Y} \tag{6}$$

Interchanging integration and summation, this can also be written as

$$C_{\text{OFDM}}(\tfrac{\sigma_a^2}{\sigma_n^2}) = \frac{1}{4^D}\sum_{i=1}^{4^D} I(\boldsymbol{A}_i;\boldsymbol{Y}) \tag{7}$$

with individual capacity[1] (mutual information) contributed by OFDM frame $\boldsymbol{A}_i$

$$I(\boldsymbol{A}_i;\boldsymbol{Y}) \overset{\text{def}}{=} \frac{1}{D}\int\!\cdots\!\int \mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{A}_i) \cdot \log_2\left(\frac{4^D\mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{A}_i)}{\sum_{l=1}^{4^D}\mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{A}_l)}\right) \mathrm{d}\boldsymbol{Y}\;. \tag{8}$$

[1]Note: $I(\boldsymbol{A}_i;\boldsymbol{Y})$ is the (normalized) *Kullback–Leibler distance* (relative entropy) [5] between $\mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{A}_i)$ and $\mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}) = \sum_{l=1}^{4^D}\frac{1}{4^D}\mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{A}_l)$. The dependency of $I(\boldsymbol{A}_i;\boldsymbol{Y})$ on σ_a^2/σ_n^2 is not explicitly stated.

For an additive noise channel, $\mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{A}) = \mathrm{f}_{\boldsymbol{N}}(\boldsymbol{Y}-\boldsymbol{A})$ holds.

When using 4-QAM modulation per carrier, it can be readily be shown that in case of no clipping the contributions $I(\boldsymbol{A}_i;\boldsymbol{Y})$ of the individual OFDM frames $\boldsymbol{A}_i$ are all identical and coincide with the capacity of 4-QAM over the AWGN channel, i.e.,

$$I(\boldsymbol{A}_i;\boldsymbol{Y}) = C_{4-\text{QAM}}(\tfrac{\sigma_a^2}{\sigma_n^2}) \tag{9}$$

hence $C_{\text{OFDM}} = C_{4-\text{QAM}}$.

B. Clipping

When clipping is active situation changes. The individual capacity of the OFDM frames $\boldsymbol{A}_i$ are no longer equal but calculate to ($\odot$: element-wise multiplication)

$$\begin{aligned} I^{(\text{CL})}(\boldsymbol{A}_i;\boldsymbol{Y}) &= I(\boldsymbol{Z}_i;\boldsymbol{Y}) = I(\alpha_i\odot\boldsymbol{A}_i;\boldsymbol{Y}) \\ &= \frac{1}{D}\int\!\cdots\!\int \mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{\alpha}_i\odot\boldsymbol{A}_i)\cdot \\ &\quad\cdot\log_2\left(\frac{4^D\mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{\alpha}_i\odot\boldsymbol{A}_i)}{\sum_{l=1}^{4^D}\mathrm{f}_{\boldsymbol{Y}}(\boldsymbol{Y}|\boldsymbol{\alpha}_l\odot\boldsymbol{A}_l)}\right)\mathrm{d}\boldsymbol{Y} \end{aligned} \tag{10}$$

It has to be noted, that the OFDM scheme would only be optimal (capacity achieving input distribution) if the individual capacities would all be equal, cf. [8, Theorem 4.5.1], [2]. This may be the starting point for an optimization.

The capacity of the OFDM scheme is then given by

$$C_{\text{OFDM}}^{(\text{CL})}(\tfrac{\sigma_a^2}{\sigma_n^2}) = \frac{1}{4^D}\sum_{i=1}^{4^D} I^{(\text{CL})}(\boldsymbol{A}_i;\boldsymbol{Y}) \tag{11}$$

which, for additive Gaussian noise can be written as ($\mathrm{h}(\cdot)$: differential entropy of given random variable or pdf)

$$= \frac{1}{D}\left(\mathrm{h}(\boldsymbol{Y}) - \mathrm{h}(\boldsymbol{N})\right) \tag{12}$$

with

$$\mathrm{h}(\boldsymbol{Y}) = -\int\!\cdots\!\int \mathrm{f}_{\boldsymbol{y}}^{(\text{CL})}(\boldsymbol{Y})\log_2\left(\mathrm{f}_{\boldsymbol{y}}^{(\text{CL})}(\boldsymbol{Y})\right)\mathrm{d}\boldsymbol{Y} \tag{13}$$

$$\mathrm{h}(\boldsymbol{N}) = D\cdot\log_2(\pi\mathrm{e}\sigma_n^2) \tag{14}$$

The pdf of the noisy channel output is thereby given by the sum of 4^D shifted (mean values $\boldsymbol{\alpha}_i\odot\boldsymbol{A}_i$) complex D-dimensional Gaussian densities. It can be written as

$$\begin{aligned} \mathrm{f}_{\boldsymbol{y}}^{(\text{CL})}(\boldsymbol{Y}) &= \frac{1}{4^D}\sum_{i=1}^{4^D}\frac{1}{(\pi\sigma_n^2)^D}\mathrm{e}^{-||\boldsymbol{Y}-\boldsymbol{\alpha}_i\odot\boldsymbol{A}_i||^2/\sigma_n^2} \\ &= \frac{1}{4^D}\sum_{i=1}^{4^D}\prod_{d=1}^{D}\frac{1}{\pi\sigma_n^2}\mathrm{e}^{-|Y_d-\alpha_{d,i}A_{d,i}|^2/\sigma_n^2} \\ &= \frac{1}{4}\sum_{i_1=1}^{4}\cdots\frac{1}{4}\sum_{i_D=1}^{4}\prod_{d=1}^{D}\frac{1}{\pi\sigma_n^2}\mathrm{e}^{-|Y_d-\alpha_{d,i}A_{d,i}|^2/\sigma_n^2}\;, \end{aligned} \tag{15}$$

with $i = 1 + (i_1 - 1) + 4(i_2 - 1) + \cdots + 4^{D-1}(i_D - 1)$. Unfortunately, unlike the unclipped situation, this pdf is not separable over the dimensions, i.e., $\mathrm{f}_{\boldsymbol{y}}^{(\mathrm{CL})}(\boldsymbol{Y}) \neq \prod_{d=1}^{D} \frac{1}{4} \sum_{i_d=1}^{4} \frac{1}{\pi\sigma_n^2} \mathrm{e}^{-|Y_d - \alpha_{d,i_d} A_{d,i_d}|^2}$; the factors α_{d,i_d} are usually different[2] for all OFDM frames (index i) and dimensions (index d).

Geometrically, in (real) $2D$ dimensional space, the set $\{\boldsymbol{A}_i\}$ of OFDM frames with 4-QAM per carrier constitute an hypercube with sides of length 2. Due to clipping, this regular arrangement is distorted—each corner point (together with its negative counterpart) is moved individually and $\{\boldsymbol{Z}_i\} = \{\boldsymbol{\alpha}_i \odot \boldsymbol{A}_i\}$ is present. This set is convolved with the (circular symmetric) Gaussian pdf resulting in $\mathrm{f}_{\boldsymbol{y}}^{(\mathrm{CL})}(\boldsymbol{Y})$.

C. Numerical Examples

We now illustrate the above derived capacities by numerical examples. Since for calculation of the capacity numerical integration over $2D$ real dimensions is required, we restrict ourselves to small D, in particular $D = 4$. Here, $4^4 = 256$ possible OFDM frames have to be assessed. However, the derived results are essentially representative for all numbers of carriers.

Figure 2 shows the individual and average capacities of 4-QAM-OFDM over the AWGN channel, $D = 4$, and $G = 2$, 1, .5 (left to right) over the signal-to-noise ratio (here: $E_\mathrm{s}/N_0 = 2/\sigma_n^2$).

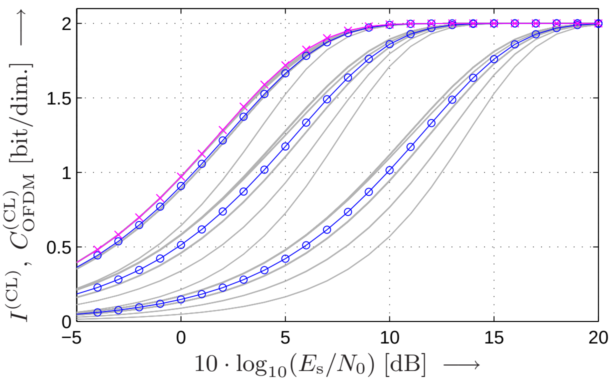

Fig. 2. Individual (gray) and average (with circles) capacities of 4-QAM-OFDM over the AWGN channel (in bits per complex dimension), $D = 4$. Left to right: clipping level $G = 2$, 1, .5. Crosses: Capacity of unclipped OFDM, equal to the capacity of 4-QAM.

For small clipping levels, the spread in the individual capacities is clearly visible. Moreover, small G lead to significant losses over unclipped OFDM.

D. Signal Shaping and Selection Metric

As the individual capacities differ if clipping is active, it is a natural idea to omit OFDM frames with low capacities from transmission. Thereby two problems arise: (i) When only the few worst frames are discarded, only very small gains can be achieved. If a large number of frames is not used, a loss in average capacity occurs (using only half of the frames, a loss of one bit per frame, hence $1/D$ bits per sample, occurs). In the optimum only some hundredth of bits can be gained. (ii) Calculation of the individual capacities is much to complex as metric for signal selection.

The second problem can be solved by the following observation. In Figure 3 scatter plots of the individual capacities $I^{(\mathrm{CL})}(\boldsymbol{A}_i; \boldsymbol{Y})$ versus the energy $||\boldsymbol{Z}_i||^2$ of the OFDM frame after clipping are given.

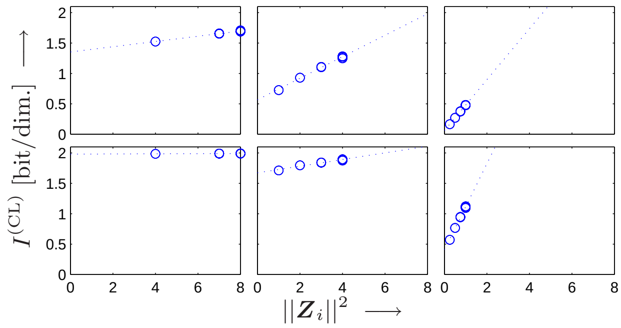

Fig. 3. Scatter plots of the individual capacities $I^{(\mathrm{CL})}(\boldsymbol{A}_i; \boldsymbol{Y})$ versus the energy $||\boldsymbol{Z}_i||^2$ of the OFDM frame after clipping. $D = 4$. Top row: $E_\mathrm{s}/N_0 \,\hat{=}\, 5$ dB, bottom row: $E_\mathrm{s}/N_0 \,\hat{=}\, 10$ dB. Left to right: clipping level $G = 2$, 1, .5. Dotted: straight-line fit.

It is immediate that capacities and energy exhibit an almost linear dependency; using other criteria, such as peak-to-average power ratio or power of the error signal, such one-to-one correspondence cannot be observed. We can hence conclude that the loss in average transmit power determines the loss in (individual) capacity. In order to increase to capacity of clipped OFDM it has hence to be aspired that the loss in energy caused by clipping is as small as possible. This can, e.g., be achieved by rotating the signals within the carriers (e.g., via a linear or quadratic, fixed phase ramp over the carriers).

IV. Capacity Bounds

The results of the last section suggest that the reduction of the transmit power due to clipping is the main source of the loss in the OFDM scheme; a result already anticipated (although not formally proven) in [14]. In order to prove this fact, we now derived upper and lower bounds on the capacity of clipped 4-QAM-OFDM.

To this end, we note that ($\mathrm{E}\{\cdot\}$: expectation)

$$\begin{aligned} \sigma_y^2 &= \frac{1}{D}\mathrm{E}\{||Y||^2\} \;=\; \frac{1}{D}(\mathrm{E}\{||Z||^2\} + \mathrm{E}\{||N||^2\}) \\ &= \frac{1}{4^D D}\sum_{i=1}^{4^D}\sum_{d=1}^{D} |\alpha_{d,i}|^2 \cdot |\pm 1 \pm \mathrm{j}|^2 + \mathrm{E}\{|N_d|^2\} \\ &= L \cdot \sigma_a^2 + \sigma_n^2 \end{aligned} \tag{16}$$

whereby the average loss factor $L \leq 1$ (over all OFDM frames) in transmit power due to clipping has been defined as

$$L \overset{\text{def}}{=} \frac{1}{4^D D}\sum_{i=1}^{4^D}\sum_{d=1}^{D} |\alpha_{d,i}|^2 \,. \tag{17}$$

Assuming Gaussian time-domain samples a_k with variance σ_a^2, i.e., $\mathrm{f}_a(a) = 1/(\pi\sigma_a^2)\mathrm{e}^{-|a|^2/\sigma_a^2}$, an approximation of this loss can readily be given. When using a soft limiter with

[2] The factors $\alpha_{d,i}$ are identical for $\boldsymbol{A}_i$ and $-\boldsymbol{A}_i$.

clipping to some amplitude limit G, transmit power calculates to

$$\sigma_z^2 = \int\cdots\int_{|a|\le G} |a|^2 \mathrm{f}_a(a)\,\mathrm{d}a + G^2 \int\cdots\int_{|a|>G} \mathrm{f}_a(a)\,\mathrm{d}a$$

which after some manipulations results in (cf. [13], [14])

$$= \sigma_a^2 \left(1 - \mathrm{e}^{-G^2/\sigma_a^2}\right) . \tag{18}$$

Hence, especially for large D where the Gaussian approximation becomes more and more accurate, the loss may be approximated by the simple expression $L_{\text{Gauss}} \overset{\text{def}}{=} 1 - \mathrm{e}^{-G^2/\sigma_a^2}$.

A. Shannon Upper Bound

In information theory, it is well known that among pdfs with the same variance, the Gaussian density has highest differential entropy [5]. Hence upper-bounding $\mathrm{h}(\boldsymbol{Y})$, with the pdf of $\boldsymbol{Y}$ given in (15), by the differential entropy of a Gaussian density we arrive at a first, simple (but loose) upper bound. Using (16) it reads

$$\begin{aligned} C^{(\text{CL})}_{\text{OFDM}}(\tfrac{\sigma_a^2}{\sigma_n^2}) &\le \log_2(\pi \mathrm{e} \sigma_y^2) - \log_2(\pi \mathrm{e} \sigma_n^2) \\ &= \log_2\left(1 + \frac{L\sigma_a^2}{\sigma_n^2}\right) \\ &= C_{\text{Shannon}}(\tfrac{L\sigma_a^2}{\sigma_n^2}) , \end{aligned} \tag{19}$$

i.e., an upper bound of the capacity of clipped OFDM is simply the Shannon AWGN capacity curve shifted by the average clipping loss.

B. QAM Upper Bound

The pdf $\mathrm{f}^{(\text{CL})}_{\boldsymbol{y}}(\boldsymbol{Y})$ is the convolution of the distorted hypercube with a (real) $2D$ dimensional Gaussian density, i.e., the sum of $4^D = 2^{2D}$ Gaussian densities with variance σ_n^2 shifted to the means $\boldsymbol{m}_i \overset{\text{def}}{=} \boldsymbol{Z}_i = \boldsymbol{\alpha}_i \odot \boldsymbol{A}_i$. Noteworthy, since $\boldsymbol{\alpha}_i$ is the scaling vector for $\boldsymbol{A}_i$ and $-\boldsymbol{A}_i$, an arrangement point symmetrical about the origin is present. We denote this sum of Gaussian pdfs as $\mathrm{f}_{\mathcal{M}}$, where the subscript indicates the dependency on the set $\mathcal{M} \overset{\text{def}}{=} \{\boldsymbol{m}_i\}$ (set of cardinality 2^{2D-1}; only one representative of $\pm\boldsymbol{m}_i$ is included).

The total energy represented by the center points $\pm\boldsymbol{m}_i$ per real dimension is $\sigma_m^2 \overset{\text{def}}{=} \sum_i ||\boldsymbol{m}_i||^2/(D2^{2D})$, which is equal to $\sigma_m^2 = \sigma_z^2/2 = L\sigma_a^2/2 = L$. We now give a (tight) upper bound on $\mathrm{h}(\mathrm{f}_{\mathcal{M}})$ by describing three consecutive operations, by which total energy (or σ_m^2 and hence σ_y^2) of the sum of Gaussians is preserved but in each step differential entropy (via variation of $\boldsymbol{m}_i$) is (slightly) increased or remains unchanged.

1) Step 1: For Gaussian densities it can be shown that the differential entropy of the sum is maximized if the Euclidean distance of their means is maximized (and hence the amount of “overlap” is minimized [11]).

Thus, keeping all but two mean values fixed, $\boldsymbol{m}_l$ (and with it $-\boldsymbol{m}_l$) is chosen to maximize $\mathrm{h}(\mathrm{f}_{\mathcal{M}})$. This is achieved by moving $\boldsymbol{m}_l$ such that it has equal distance to all its $2D$ nearest neighboring points $\boldsymbol{m}_i$ while keeping $||\boldsymbol{m}_l||$ (and hence total energy) constant (moving $\boldsymbol{m}_l$ on the surface of a $2D$ dimensional hypersphere centered at the origin).

This procedure is repeated for all points $\boldsymbol{m}_i$ and iterated until convergence is reached. At each step, since minimum Euclidean distance is increased, differential entropy is increased, too.

In the end, all point have the same Euclidean distance from their neighboring points (otherwise improvements by further iterations would be possible); a hyperparallelepiped centered at the origin and with edges of equal length results. Since energy is kept constant during this procedure, the length of the edges is $2\sigma_m$.

2) Step 2: In the next step, the parallelepiped is compressed along its longest body diagonal. While keeping the side lengths constant and preserving point symmetry wrt. the origin, the parallelepiped is transformed into a hypercube. Thereby, energy is not changed (which follows from the Pythagorean theorem). But since the distances to the second- and higher-order neighbors are increased, differential entropy is again (slightly) increased.

3) Step 3: In the last step, without changing entropy, via a unitary transformation (pure rotation) the hypercube can be aligned such that its sides are parallel to the coordinate axis.

Hence, we arrive at a hypercube whose vertices are given by the Cartesian product $\{\pm\sigma_m\}^{2D} = \{\pm\sqrt{L}\}^{2D}$. It is guaranteed that this arrangement has the same energy as the initial (irregular) one but (somewhat) higher differential entropy.

Figure 4 visualizes the maximization of the differential entropy of the sum of 4 Gaussians in 2 dimensional space. Additionally, the above approximation via a Gaussian density with the same variance is shown. The tightness of the new bound (in contrast to the Gaussian one) is evident.

In summary, $\mathrm{f}^{(\text{CL})}_{\boldsymbol{y}}(\boldsymbol{Y})$ has a lower differential entropy than the convolution of a regular (separable over the dimensions) hypercube and a Gaussian density. The hypercube is thereby adjusted to have the same average energy as the distorted one; the signal points in each complex dimension are then taken from

$$\sqrt{L}\mathcal{A}$$

Hence, differential entropy is upper bounded by

$$\mathrm{h}(\boldsymbol{Y}) \le \mathrm{h}(\boldsymbol{Y}^{\square}) \tag{20}$$

with pdf

$$\mathrm{f}_{\boldsymbol{y}}(\boldsymbol{Y}^{\square}) = \prod_{d=1}^{D} \frac{1}{4} \sum_{i_d=1}^{4} \frac{1}{\pi\sigma_n^2} \mathrm{e}^{-|Y_d - \sqrt{L}A_{d,i_d}|^2/\sigma_n^2} . \tag{21}$$

Due the separability over the dimension, as in the case of the unclipped AWGN channel, this entropy readily calculates to

$$\mathrm{h}(\boldsymbol{Y}^{\square}) = D \cdot \mathrm{h}(Y^{(\text{4-QAM scaled by } \sqrt{L})}) \tag{22}$$

In summary, we arrive at the main result

$$\begin{aligned} C^{(\text{CL})}_{\text{OFDM}}(\tfrac{\sigma_a^2}{\sigma_n^2}) &= \frac{1}{D}\left(\mathrm{h}(\boldsymbol{Y}) - \mathrm{h}(\boldsymbol{N})\right) \\ &\le \mathrm{h}(Y^{(\text{4-QAM scaled by } \sqrt{L})}) - \frac{1}{D}\mathrm{h}(\boldsymbol{N}) \\ &= C_{4-\text{QAM}}(\tfrac{L\sigma_a^2}{\sigma_n^2}) = C_{\text{OFDM}}(\tfrac{L\sigma_a^2}{\sigma_n^2}) \end{aligned} \tag{23}$$

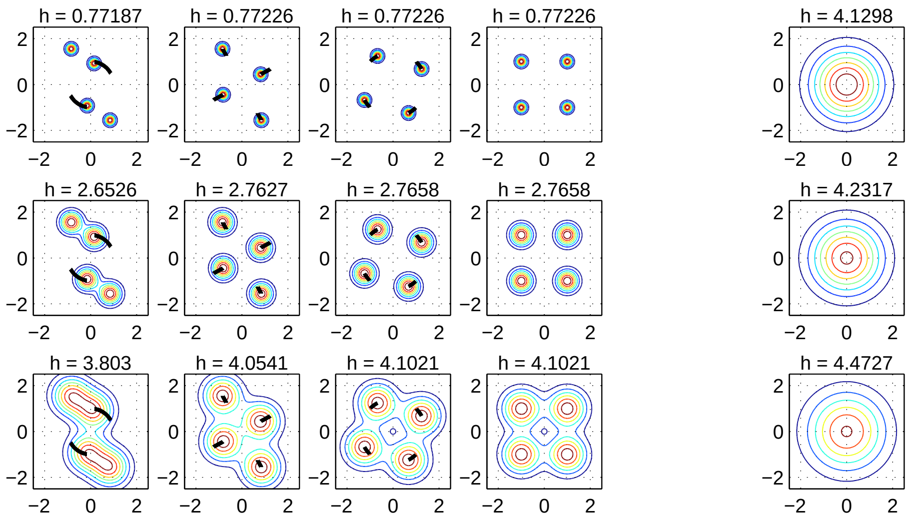

Fig. 4. Visualization of the maximization of the differential entropy of the sum of Gaussians (2 dimensional example; 4 signal points; contour plot of Gaussian densities). Rows: high to low SNR. Left to right: initial constellation; rotation of two points; compression towards square; rotation of all points for aligning with the coordinates. The movement of the Gaussian densities is indicated by the solid black lines. $\sigma_m^2 = 1$. Rightmost plots: Gaussian density with the same variance σ_y^2.

In words, an upper bound on the capacity of clipped 4-QAM-OFDM is simply given by the capacity of the unclipped scheme, operating at a signal-to-noise ratio lowered by the loss factor L; the capacity curve is simply shifted by the loss factor L towards higher SNRs.

We note that this upper bound is tight for both, low and high SNRs. At low SNR, $\boldsymbol{Y}$ tend to be Gaussian anyway; the arrangement of the points does not matter. At high SNRs, the arrangement of the points does not matter either, as the Gaussian densities are separated and $\mathrm{h}(\boldsymbol{Y})$ converges to $D\log_2(4) + D\log_2(\pi \mathrm{e} \sigma_n^2)$.

Since "nearest neighbor" OFDM frames $\boldsymbol{A}_i$ and $\boldsymbol{A}_l$ differ only in one position, the time-domain signals will be not too different (especially when D is large). In turn, they will be affected similarly by clipping. Hence the corresponding scaling factors $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_l$ will be very similar. Since this holds for all nearest neighbors of an OFDM frame, $\boldsymbol{Z}_i$ will almost have the same distance from its nearest neighbors in $2D$ dimensional real space. The initial hypercube with vertices $\{\boldsymbol{A}_i\}$ is distorted, but attenuation will be the dominating effect over deformation. Hence, it can be expected that the upper bound on $\mathrm{h}(\boldsymbol{Y})$ is very tight over the entire region of SNR.

C. Lower Bound

The 4-QAM capacity curve is a strictly concave function over the SNR (in linear scale). Using Jensen's inequality, this leads to

$$C_{4-\mathrm{QAM}}(L\tfrac{\sigma_a^2}{\sigma_n^2}) = C_{4-\mathrm{QAM}}\left(\frac{1}{4^D D}\sum_{i=1}^{4^D}\sum_{d=1}^{D} |\alpha_{d,i}|^2 \tfrac{\sigma_a^2}{\sigma_n^2}\right) \geq \frac{1}{4^D D}\sum_{i=1}^{4^D}\sum_{d=1}^{D} C_{4-\mathrm{QAM}}\left(|\alpha_{d,i}|^2 \tfrac{\sigma_a^2}{\sigma_n^2}\right)$$

and since the sets $\{\alpha_{d,i} | d \text{ given}, i = 1, \ldots, M^D\}$ are the same for all d, we have

$$= \frac{1}{4^D}\sum_{i=1}^{4^D} C_{4-\mathrm{QAM}}\left(|\alpha_{d,i}|^2 \tfrac{\sigma_a^2}{\sigma_n^2}\right) \ . \tag{24}$$

From information theory it is known that treating the carriers separately, a lower bound on the capacity is established, i.e., $I(\boldsymbol{A};\boldsymbol{Y}) \geq \sum_d I(A_d;Y_d)$. This is the case in the above equation: the individual capacities the carriers are summed up (and normalized to the number D of carriers). Transmission of an individual OFDM frame $\boldsymbol{A}_i$ leads to a specific "fading amplitude" $\alpha_{d,i}$. When knowing this factor at the receiver, capacity is given by the average of the individual capacities given $\alpha_{d,i}$ is active. Equation (24) is hence a lower bound on the capacity, assuming $\alpha_{d,i}$ is known to the receiver.

At this point it has to be noted that achieving $C_{\mathrm{OFDM}}^{(\mathrm{CL})}$ or its lower bound (24) requires the knowledge of the "channel state" $\boldsymbol{\alpha}_i$. This vector, however, is one-to-one related to the initial OFDM frame $\boldsymbol{A}_i$. When estimating the vectors $\boldsymbol{Z}_i$, data $\boldsymbol{A}_i$ and induced channel state $\boldsymbol{\alpha}_i$ are treated jointly and the derived capacity is indeed achievable. For realizing (24), carrier-wise processing is possible but the channel state has to be known. This, however, is only possible if the entire frame is considered. Approaches for treating clipped OFDM frames and their reconstruction are given, e.g., in [9], [17], [3], [15].

D. Statistical Clipper Model

In [13], [14], a statistical model for describing the clipping nonlinearity has been used. Let $\gamma = G/\sigma_a$ be the clipping ratio, then the clipper output is expressed as

$$z_k = g(a_k) = \eta a_k + d_k \tag{25}$$

where

$$\eta = 1 - \mathrm{e}^{-\gamma^2} + \gamma\frac{\sqrt{\pi}}{2}\mathrm{erfc}(\gamma) \tag{26}$$

is a "constellation shrinking factor" and d_k is approximated by white Gaussian noise, uncorrelated with a_k, and with variance

$$\sigma_d^2 = \sigma_a^2\left(1 - \mathrm{e}^{-\gamma^2} - \eta^2\right) \ . \tag{27}$$

In frequency domain, $Y_d = \eta A_d + D_d + N_d$ results, and since D_d and N_d are independent noise terms, a lower bound on the capacity of clipped OFDM reads

$$C^{(\mathrm{CL})}_{\mathrm{OFDM}}(\tfrac{\sigma_a^2}{\sigma_n^2}) \geq C_{\mathrm{statist}}(\tfrac{\sigma_a^2}{\sigma_n^2}) \overset{\mathrm{def}}{=} C_{4-\mathrm{QAM}}(\tfrac{\eta^2\sigma_a^2}{\sigma_n^2+\sigma_d^2}) \,. \qquad (28)$$

Assuming Gaussian signaling within the carriers, $\log_2(1 + \eta^2\sigma_a^2/(\sigma_n^2+\sigma_d^2))$ holds [14].

E. Numerical Results

We illustrate the derived capacities bounds via numerical examples. Again, $D = 4$ and transmission over the AWGN channel is assumed. Figure 5 shows the (numerically evaluated) average capacities of 4-QAM-OFDM over the AWGN channel together with the Shannon and the QAM upper bounds, the lower bound, and the statistical model for $G = 2$, 1, .5 (left to right) over the signal-to-noise ratio.

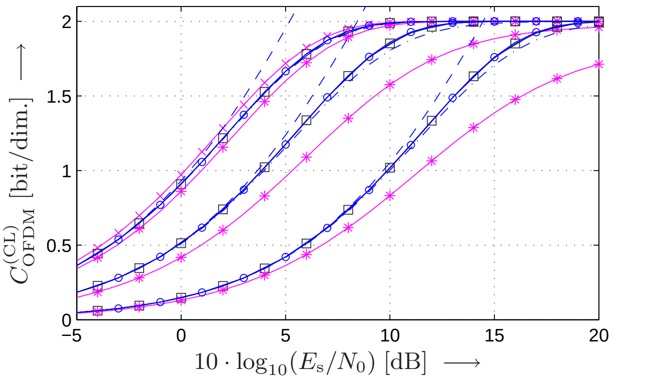

Fig. 5. Average capacities of 4-QAM-OFDM over the AWGN channel (circles) together with the Shannon (dashed) and the QAM (squares) upper bounds, the lower bound (dash-dotted), and the statistical model (stars) for $G = 2, 1, .5$. $D = 4$. Crosses: Capacity of unclipped OFDM, equal to the capacity of 4-QAM.

It can be seen that the derived QAM upper bound (4-QAM curve shifted by the power loss) is very tight and cannot be distinguished from the numerical evaluation of the capacity. The Shannon upper bound becomes loose for large SNR, whereas the lower bound is very tight for all SNRs. The capacity curve given by the statistical model is much too pessimistic; the effect of clipping is treated as additional noise, independent over the carriers, whereas the other bounds utilize the correlations over frequency.

V. Concluding Remarks and ISI channels

In this paper, we have studied the capacity of clipped 4-QAM-OFDM. Central result is a very tight upper bound on the capacity, which is simply the 4-QAM capacity curve shifted (towards higher SNRs) by the power loss induced by clipping. This capacity can be utilized if joint ML decoding of the entire OFDM frame is performed. Respective algorithms are discussed in literature, e.g., [17], [3], [15].

In this paper, due to lack of space, only transmission over the AWGN channel is studied. However, assuming channel knowledge at the receiver, the results are also valid for intersymbol-interference channels, which transform to (almost) Rayleigh-fading channels over the carriers. Replacing the factors α_d by $H_d\alpha_d$, all above equations hold for a particular channel with given impulse response/transfer function. Averaging over the capacities according to the Rayleigh distribution gives the desired result, cf. [13].

Since the power loss due to clipping is the major source of losses, optimization of the OFDM scheme aiming at higher power at the output of the clipper is obvious. One approach would be to deviate from a fixed signal grid over the carriers, but rotate each carrier individually, e.g., by adding a linear or quadratic phase ramp. Contrary to this fixed signal operation—each possible OFDM frame is modified equally—selection methods similar to peak-to-average power ratio reduction are imaginable, cf. [12], [18].

Finally, the results should be extended to higher modulation alphabets. As long as clipping does not destroy the one-to-one mapping of information to clipped frames (cf. the discussion in [14]), similar statements as for 4-QAM are expected.

References

[1] J.A.C. Bingham. Multicarrier Modulation for Data Transmission: An Idea Whose Time Has Come. *IEEE Comm. Mag.*, pp. 5–14, May 1990.

[2] R.E. Balhut. Computation of Channel Capacity and Rate Distortion Functions. *IEEE Tr. Inf. Theory*, pp. 460–473, July 1972.

[3] H. Chen, A.M. Haimovich. Iterative Estimation and Cancellation of Clipping Noise for OFDM Signals. *IEEE Comm. Letters*, pp. 305–307, July 2003.

[4] E. Costa, S. Pupolin. M-QAM-OFDM Systems Performance in the presence of a Nonlinear Amplifier and Phase Noise. *IEEE Tr. Comm.*, pp. 462–472, March 2002.

[5] T.M. Cover, J.A. Thomas. *Elements of Information Theory*. John Wiley & Sons, Inc., New York, 1991.

[6] D. Dardari, V. Tralli, A. Vaccari. A Theoretical Characterization of Nonlinear Distortion Effects in OFDM Systems. *IEEE Tr. Comm.*, pp. 1755–1764, Oct. 2000.

[7] R. Déjardin, M. Colas, G. Gelle. On the Iterative Mitigation of Clipping Noise for COFDM transmission. *European Tr. on Telecomm. (ETT)*, pp. 791–800, Sept. 2008.

[8] R.G. Gallager. *Information Theory and Reliable Communication*. John Wiley & Sons, Inc., New York, London, 1968.

[9] D. Kim, G.L. Stüber. Clipping Noise Mitigation for OFDM by Decision-Aided Reconstruction. *IEEE Comm. Letters*, pp. 4–6, Jan. 1999.

[10] M. Krondorf, G. Fettweis. OFDM Link Performance Analysis under Various Receiver Impairments. *EURASIP J. Wireless Comm. and Networking*, Article ID 145279, 2008.

[11] S. Kullback. A Lower Bound for Discrimination Information in Terms of Variation. *IEEE Tr. Inf. Theory*, pp. 126–127, Jan. 1967.

[12] X. Lei, Y. Tang, S. Li, Y. Li. A Minimum Clipping Power Loss Scheme for Mitigating the Clipping Noise in OFDM. *Proc. Global Telecomm. Conf. (GLOBECOM '03)*, San Francisco, USA, pp. 6–9, Dec. 2003.

[13] H. Ochiai, H. Imai. Performance Analysis of Deliberately Clipped OFDM Signals. *IEEE Tr. Comm.*, pp. 89–101, Jan. 2002.

[14] F. Peng, W.E. Ryan. On the Capacity of Clipped OFDM Channels. *Proc. Int. Symp. Inf. Theory (ISIT) 2006*, Seattle, USA, July 2006.

[15] F. Peng, W.E. Ryan. MLSD Bounds and Receiver Design for Clipped OFDM Channels. *IEEE Tr. Comm.*, pp. 3568–3578, Sept. 2008.

[16] J.G. Proakis. *Digital Communications*. McGraw-Hill, New York, 2001.

[17] J. Tellado, L.M.C. Hoo, J.M. Cioffi. Maximum-Likelihood Detection of Nonlinearly Distorted Multicarrier Symbols by Iterative Decoding. *IEEE Tr. Comm.*, pp. 218–228, Feb. 2003.

[18] G.R. Tsouri, D. Wulich. Capacity Analysis and Optimisation of OFDM with Distortionless PAPR Reduction. *European Tr. on Telecomm. (ETT)*, pp. 781–790, Sept. 2008.