On the Limits of Coded Transmission over Fading Channels with CDMA

Alexander Lampe, and Johannes B. Huber
Telecommunications Institute II, University of Erlangen
Cauerstraße 7/NT, D–91058 Erlangen, Germany
Phone: +49-9131-85 27114, Fax: +49-9131-85 28919, Email: alampe@lnt.de

Abstract — **In this paper we investigate the synchronous transmission over time–variant multipath Rayleigh–fading channels employing Direct Sequence Code Division Multiple Access (DS–CDMA). Assuming ideal knowledge of the actual channel state, we introduce an equivalent transmission model for the case of high processing gain. Based on this model an analytical solution for the spectral efficiency achievable by application of various nonlinear receivers can be given. Here, we consider the application of linear interference suppression by means of MMSE filters combined with successive cancellation and single user decoding. We show that in this way the system's spectral efficiency approaches that of the AWGN channel if the number of users K is much larger than the spreading factor N. This holds for the flat Rayleigh fading channel as well as for frequency selective channels with a large number of propagation paths.**

I. Introduction

Recently, the search for information theoretical bounds regarding the transmission of K users with code division multiple access (CDMA) to a single receiver has attracted considerable attention. So, in [1, 2] the system's capacity resulting for several linear multiuser receivers with and without decision feedback was derived for synchronous transmission over additive white Gaussian noise (AWGN) channels. Further, for transmission over attenuated single path channels with constant path gains which are known at the receiver, it was shown that linear interference suppression by means of an MMSE filter combined with single user decoding and successive interference cancellation achieves the same capacity as the overall joint optimum decoder (see [3, 2, 4]). Next, assuming a random distribution of the users' powers as well as random spreading sequences it was shown by Tse and Hanly [5] that the signal to noise ratio at the output of a linear MMSE filter reaches a nonrandom limit under the condition of infinite processing gain and constant system load $\beta = K/N$. This result was derived from the distribution of eigenvalues of large random covariance matrices (see also [6]).

In this work we address the problem of reliable transmission over frequency selective fading channels with synchronous CDMA. Taking into account the practical relevance of large spreading factors for forthcoming communications systems an equivalent model for the synchronous transmission over multipath fading channels with randomly chosen spreading sequences is derived for the limit $N \to \infty$. Based on this model and assuming perfect channel state information a closed solution on the system's spectral efficiency achievable with MMSE interference suppression and successive interference cancellation (MMSE–SIC) is given. With this analytical result, the influence of the system load β and the number of propagation paths L on the spectral efficiency is studied. In addition, we show that the spectral efficiency of the optimal joint decoder can be achieved by MMSE–SIC in the case of fading channels with known path gains, too.

The paper is arranged as follows. In Section II, the actual and the equivalent transmission model for high processing gain are given. Based on the equivalent model, the capacity for CDMA employing MMSE–SIC at the receiver is derived in Section III. Finally, Section IV points out conclusions.

II. Transmission Models

We consider the transmission of K users over independent frequency selective channels with CDMA to a single receiver. Assuming that all users transmit synchronously the underlying discrete–time equivalent complex baseband transmission model is illustrated in Fig. 1a.

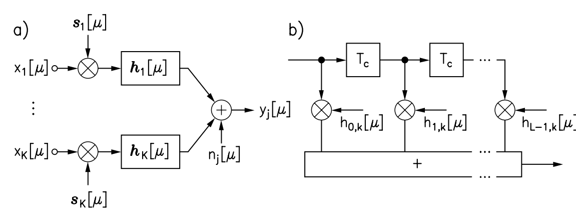

Fig. 1: a) Transmission model of CDMA system with K users b) Tapped–delay–line channel model with L paths.

Here, $x_k[\mu]$ and $\boldsymbol{s}_k[\mu] = (s_{1,k}[\mu], \ldots, s_{N,k}[\mu])^T$ denote the kth user's transmitted channel symbol and his/her spreading sequence in the μth transmission interval, respectively. The users' channel symbols are chosen from the set $x_k[\mu] \in \mathcal{X}$ and the elements of the unit energy spreading sequence are drawn randomly as $s_{j,k}[\mu] \in \{(\pm 1 \pm \mathrm{j})/\sqrt{2N}\}, \forall j$. The kth user's modulated sequence $(x_k[\mu]s_{1,k}[\mu], \ldots, x_k[\mu]s_{N,k}[\mu])$ is transmitted over a channel with time variant weight function $\boldsymbol{h}_k[\mu]$. In order to account for the multipath propagation we use the well known tapped–delay–line channel model (see Fig. 1b) assuming L resolvable paths. T_c is the length of one chip interval. As usual we assume the path gains $h_{0,k}[\mu], \ldots, h_{L-1,k}[\mu], \forall k$, to be independent and identically zero mean proper complex Gaussian distributed with variance $\sigma_{l,h}^2 = \mathcal{E}\{|h_{l,k}|^2\} = 1/L$, $\forall l$, $\forall k$. Thus, in order to simplify the derivations we restrict our considerations to so–called equal gain channels.[1] Supposing the path weights to be constant over one transmission interval the received signal resulting from $x_k[\mu]$ can be written as $(x_k[\mu]s'_{1,k}[\mu], \ldots, x_k[\mu]s'_{N',k}[\mu])$, where the kth user's effective spreading sequence $\boldsymbol{s}'_k[\mu] = (s'_{1,k}[\mu], \ldots, s'_{N',k}[\mu])^T$ results from the convolution of the actual spreading sequence $\boldsymbol{s}_k[\mu]$ and the channel's impulse response $\boldsymbol{h}_k[\mu]$, i.e., $s'_{j,k}[\mu] = \sum_{\eta=1}^{N} s_{\eta,k}[\mu] h_{(j-\eta),k}[\mu]$.

In the following we look at the behavior of this multiple access system in the case of large spreading factors. In fact, we consider the situation $N \gg 1$ while keeping the physical bandwidth allocated to each user constant, what means that the data symbol interval is considerably longer than the chip duration T_c. Of course, this implies that the data rate of each single user is relatively small compared to total system rate for sufficiently large load β. However, it is to be expected that the data rates required in the uplink of future mobile CDMA systems will be much lower than in the downlink. So, this assumption can be used to evaluate analytically the system's performance describing the uplink scenario in mobile telecommunication systems.

But, before we can proceed with that calculation we have to simplify our transmission model first. For this, we consider the μth transmission interval.

The average expected power σ_{ISI}^2 of the intersymbol interference (ISI) due to a single user's transmission in the $(\mu-1)$st as well as $(\mu+1)$st interval affecting the μth data symbol interval of interest is obtained as

$$\sigma_{\mathrm{ISI}}^2 = \frac{2}{L}\sigma_x^2 \sum_{\lambda=0}^{L-1} \frac{\lambda}{N} = \sigma_x^2 \frac{(L-1)}{N}. \qquad (1)$$

Next, with a finite number of propagation paths [2] we have that the ratio L/N approaches zero for rising N. Thus, the power of the intersymbol interference resulting from each user goes to zero. On the other hand, each user's signal energy allocated to one transmission interval is constant due to the choice of unit energy spreading sequences. So, supposing for the system load $\beta = \text{constant}$ the dominating interference is caused by channel noise as well as multiuser interference. As a consequence, the characteristic properties of the system under study remain unchanged by neglecting the ISI for $N \gg L$. Now, arranging the signals received at chip rate in the μth interval in the vector $\boldsymbol{y}[\mu] = (y_1[\mu], \ldots, y_{N'}[\mu])^T$ leads to

$$\boldsymbol{y}[\mu] \;=\; \boldsymbol{S'}[\mu]\boldsymbol{x}[\mu] + \boldsymbol{n}[\mu]. \qquad (2)$$

[1] $\mathcal{E}\{x\}$ and $|x|$ denote the expectation value and the absolute magnitude of x, respectively.

[2] Note, that the number of paths needed to be resolved in this system is limited since T_c is fixed.

The $N' = N + L - 1$ dimensional vector $\boldsymbol{n}[\mu] = (n_1[\mu], \ldots, n_{N'}[\mu])^T$ represents the additive channel noise, where the i.i.d. samples $n_j[\mu], 1 \le j \le N'$, are zero mean complex Gaussian variables with variance σ_n^2. Further, $\boldsymbol{x}[\mu] = (x_1[\mu], \ldots, x_K[\mu])^T$ and $\boldsymbol{S'}[\mu] = (\boldsymbol{s}'_1[\mu], \ldots, \boldsymbol{s}'_K[\mu])$ consist of the K users' transmitted symbols and their effective spreading sequences as well.

It can be seen that the ratio of each user's signal energy contained in the first $L-1$ as well as last $L-1$ samples of the received signal $\boldsymbol{y}$ to the whole energy transmitted by each user in one symbol interval tends to zero for rising N. In addition, the same holds for the energy of the channel noise while the relation between the total channel noise power and signal power as well as the multiuser interference in each transmission interval remains unchanged.

So, focusing on the elements $s'_{j,k}[\mu], j = L, \ldots, N, \forall \mu$, we find that their distribution (and hereby the system's spectral efficiency) is not changed for $N \gg 1$ if each element is randomly chosen as

$$s'_{j,k}[\mu] \;\triangleq\; \sum_{\lambda=0}^{L-1} h_{k,\lambda}[\mu] s_{j,k,\lambda}[\mu],\; j = L, \ldots, N, \qquad (3)$$

where $h_{k,\lambda}[\cdot]$ is zero complex Gaussian distributed with variance $1/L$ and $s_{j,k,\lambda}[\cdot] \in \{(\pm 1 \pm \mathrm{j})/\sqrt{2N}\}$. This parallels the result recently given in [7] stating that for infinite processing gain the shifted replicas of one user's spreading sequence arriving at the receiver over the L propagation paths are equivalent to L different randomly chosen spreading sequences having unit norm. Moreover, regarding the conditions for the eigenvalue distribution of random covariance matrices given in [6] it can be shown that the results given in the next section can also be obtained by modeling the effective spreading sequence as product $\boldsymbol{s}'[\mu] \triangleq a_{k,L}[\mu]\hat{\boldsymbol{s}}_k[\mu]$. Here, the square of the scalar $a_{k,L}[\mu] \in \mathbb{R}_0^+$ stands for the power of the right hand side of Eq. 3 being normalized by $1/N$, i.e., $a_{k,L}^2[\mu] \widehat{=} \sum_{\lambda=0}^{L-1} |h_{k,\lambda}[\mu]|^2$. So, the probability density function of $a_{k,L}^2[\cdot], \forall k$, is given as [8]

$$f_{a_L^2}(u) \;=\; \frac{L^L}{(L-1)!} u^{(L-1)} \mathrm{e}^{-Lu} \delta_{-1}(u). \qquad (4)$$

Here, $\delta_{-1}(u)$ denotes the unit step function. Further, the independent and identically distributed elements of $\hat{\boldsymbol{s}}_k[\mu]$ can be randomly chosen from any arbitrary complex distribution with zero mean and variance $1/N$. So, for $N \to \infty$ we end up with the model

$$\boldsymbol{y}[\mu] \;=\; \hat{\boldsymbol{S}}[\mu]\boldsymbol{A}[\mu]\boldsymbol{x}[\mu] + \boldsymbol{n}[\mu]. \qquad (5)$$

Note it is still assumed that $\hat{\boldsymbol{S}}[\mu] = (\,\hat{\boldsymbol{s}}_k[\mu]\,)$ and $\boldsymbol{A}[\mu] = \mathrm{diag}(\,a_{k,L}[\mu]\,)$ are perfectly known to the receiver in each transmission interval. Further, it should be stressed that this model is valid even for finite processing gains in the cases of synchronous transmission over a flat Rayleigh fading channel as well as the additive white Gaussian noise channel.

III. Capacity of CDMA applying MMSE–SIC

In order to maximize the mutual information $I(\boldsymbol{y};\boldsymbol{x})$ of the considered multiple access system all channel symbols have to be drawn i.i.d. from a Gaussian distribution with equal power, i.e., $\mathcal{E}\{|x_k[\mu]|^2\} = \sigma_x^2$. We investigate the spectral efficiency achievable by application of appropriately chosen MMSE filters in conjunction with successive cancellation and single user decoding at the receiver. For this, we first determine the signal to noise ratio (SNR) provided at the output of an MMSE filter for a specific user, supposing that the number of uncancelled interfering users is $\Upsilon - 1$.

Let us consider a system with Υ users defining user Υ as user of interest. Then, the signal to noise ratio $\mathrm{SNR}_\Upsilon[\mu]$ in the μth transmission interval at the output of an MMSE–filter adapted to user Υ is [9]

$$\begin{aligned}\mathrm{SNR}_\Upsilon[\mu] \quad &= \\ a^2_{\Upsilon,L}[\mu]\hat{\boldsymbol{s}}^H_\Upsilon[\mu] &\left(\sum_{k=1}^{\Upsilon-1}\hat{\boldsymbol{s}}_k[\mu]a^2_{k,L}[\mu]\hat{\boldsymbol{s}}^H_k[\mu]+\frac{\sigma_n^2}{\sigma_x^2}\boldsymbol{I}\right)^{-1}\hat{\boldsymbol{s}}_\Upsilon[\mu] \\ \triangleq a^2_{\Upsilon,L}[\mu]\hat{\boldsymbol{s}}^H_\Upsilon[\mu] &\left(\hat{\boldsymbol{S}}_\Upsilon[\mu]\boldsymbol{A}^2_\Upsilon[\mu]\hat{\boldsymbol{S}}^H_\Upsilon[\mu] + \frac{\sigma_n^2}{\sigma_x^2}\boldsymbol{I}\right)^{-1}\hat{\boldsymbol{s}}_\Upsilon[\mu],\end{aligned}$$

where $\boldsymbol{I}$ denotes the identity matrix and H the conjugate transpose. Now, for $N\to\infty$ (implying $\Upsilon\to\infty$) and regarding that the users amplitudes $a_{k,L}[\mu], \forall k$, are drawn from identical wide sense stationary random processes the empirical distribution of the eigenvalues of $\mathrm{diag}\left(a^2_{k,L}[\mu]\right)$ converges for each μ to $f_{a^2_L}(u)$. So, we get with Theorem 3.1 of [5] for the signal to noise ratio of user Υ (see also [6])

$$\lim_{N\to\infty}\mathrm{SNR}_\Upsilon[\mu] = \mathrm{SNR}^{\lim}_\Upsilon[\mu] \tag{6}$$

$$= \frac{a^2_{\Upsilon,L}[\mu]}{\sigma_n^2/\sigma_x^2 + \alpha\int\limits_0^\infty \frac{a^2_{\Upsilon,L}[\mu]\zeta}{a^2_{\Upsilon,L}[\mu]+\mathrm{SNR}^{\lim}_\Upsilon[\mu]\zeta} f_{a^2_L}(\zeta)\mathrm{d}\zeta}. \tag{7}$$

Introducing the normalized signal to noise ratio depending on the ratio $\alpha = \Upsilon/N, \alpha = \mathrm{const.}$ and being denoted as $\overline{\mathrm{SNR}}^{\lim}(\alpha) \triangleq \mathrm{SNR}^{\lim}_\Upsilon[\mu]/a^2_{\Upsilon,L}[\mu]$ the above condition becomes independent of the specific time index μ and reads

$$\overline{\mathrm{SNR}}^{\lim}(\alpha) = \frac{1}{\sigma_n^2/\sigma_x^2 + \frac{\alpha}{\overline{\mathrm{SNR}}^{\lim}(\alpha)}\int\limits_0^\infty \frac{\zeta}{1/\overline{\mathrm{SNR}}^{\lim}(\alpha)+\zeta} f_{a^2_L}(\zeta)\mathrm{d}\zeta}. \tag{8}$$

The integral in the denominator of the above expression can be solved analytically for a given number of paths L and yields

$$\int\limits_0^\infty \frac{\zeta}{1/\overline{\mathrm{SNR}}^{\lim}(\alpha)+\zeta} f_{a^2_L}(\zeta)\mathrm{d}\zeta = \int\limits_0^\infty \frac{\zeta}{1/\overline{\mathrm{SNR}}^{\lim}(\alpha)+\zeta}\ \frac{L^L}{(L-1)!}\zeta^{L-1}\mathrm{e}^{-L\zeta}\mathrm{d}\zeta \tag{9}$$

$$= \frac{(\overline{\mathrm{SNR}}^{\lim}(\alpha))^{-L}\mathrm{e}^{L/\overline{\mathrm{SNR}}^{\lim}(\alpha)}\mathrm{Ei}\left(\frac{-L}{\overline{\mathrm{SNR}}^{\lim}(\alpha)}\right)}{(-1)^{1-L}L^{-L}(L-1)!} + \frac{\sum\limits_{\lambda=1}^{L}(\lambda-1)!(-\overline{\mathrm{SNR}}^{\lim}(\alpha))^{\lambda-L}L^{-\lambda}}{L^{-L}(L-1)!}, \tag{10}$$

where $\mathrm{Ei}(\zeta) \triangleq -\int\limits_{-\zeta}^{\infty}\mathrm{e}^{-\tau}\tau^{-1}\mathrm{d}\tau$. As Eq. (8) has a unique fixed point (see [5]) it is possible to solve $\overline{\mathrm{SNR}}^{\lim}(\alpha)$ iteratively. Equipped with $\overline{\mathrm{SNR}}^{\lim}(\alpha)$ and regarding that the users' channel symbols are independent and identically Gaussian distributed user Υ can transmit arbitrarily reliable at rate R_Υ as long as $R_\Upsilon < C_\Upsilon$ where the capacity C_Υ is [10]

$$C_\Upsilon = \mathop{\mathcal{E}}_{\mu}\left\{\log 2\left(1+\overline{\mathrm{SNR}}^{\lim}(\alpha)\, a^2_{\Upsilon,L}[\mu]\right)\right\} \tag{11}$$

$$\begin{aligned}&= \frac{L^L}{(L-1)!}\int\limits_0^\infty \log 2\left(1+\overline{\mathrm{SNR}}^{\lim}(\alpha)\zeta\right)\frac{\mathrm{e}^{-\zeta L}}{\zeta^{1-L}}\mathrm{d}\zeta \\ &= \frac{(-1)^L}{\ln(2)(L-1)!}\,\frac{\partial^{L-1}}{\partial\theta^{L-1}}\left[\frac{-1}{\theta}\exp\left(\frac{L\theta}{\overline{\mathrm{SNR}}^{\lim}(\alpha)}\right)\right. \\ &\quad \left.\times\mathrm{Ei}\left(-L\theta(\overline{\mathrm{SNR}}^{\lim}(\alpha))^{-1}\right)\right]_{\theta=1}.\end{aligned} \tag{12}$$

Note, that the capacity C_Υ relies on the ratio α determining the number of interfering users in multiples of the processing gain N. So, regarding that for the whole system holds $\beta = K/N$ and indexing the users in the reverse order they are successively decoded and cancelled it follows for the kth user with $k = \gamma K, \gamma \in (0,1]$ that $\alpha = k/N = \beta\cdot\gamma$. In other words, in the limit $N\to\infty$ we have $\alpha\in(0,\beta]$. Thus, the spectral efficiency of the considered CDMA system is solved as

$$\varGamma^{\lim}_{\mathrm{sys}} \triangleq \lim_{N=K/\beta\to\infty}\frac{1}{N}\sum_{k=1}^{K}C_k = \int\limits_0^\beta C(\alpha)\mathrm{d}\alpha, \tag{13}$$

where $C(\alpha) \triangleq C_k$.

To illustrate the above result we consider the case $L=1$, i.e., the single path Rayleigh fading channel and the case $L \gg 1, L/N \approx 0$, which is well–known to be equivalent to the transmission over an AWGN channel.

First, for $L=1$ Eq. (10) simplifies to

$$\int\limits_0^\infty \frac{\zeta}{1/\overline{\mathrm{SNR}}^{\lim}(\alpha)+\zeta}\exp(-\zeta)\mathrm{d}\zeta = \frac{\mathrm{e}^{1/\overline{\mathrm{SNR}}^{\lim}(\alpha)}}{\overline{\mathrm{SNR}}^{\lim}(\alpha)}\mathrm{Ei}\left(-1/\overline{\mathrm{SNR}}^{\lim}(\alpha)\right)+1, \tag{14}$$

whereas for $L \gg 1$ using $f_{a^2_L}(\zeta) \overset{L\gg 1}{\approx} \delta(\zeta-1)$, ($\delta(.)$ denotes the Dirac impulse) is solved

$$\int\limits_0^\infty \frac{\zeta}{1/\overline{\mathrm{SNR}}^{\lim}(\alpha)+\zeta} f_{a^2_L}(\zeta)\mathrm{d}\zeta = \frac{\overline{\mathrm{SNR}}^{\lim}(\alpha)}{1+\overline{\mathrm{SNR}}^{\lim}(\alpha)}. \tag{15}$$

Thus, for $L \gg 1$ the required SNR can be obtained explicitly only relying on the load α and the signal as well as noise power σ_x^2 and σ_n^2, respectively (see also [5], [1])

$$\overline{\mathrm{SNR}}^{\lim}(\alpha) = -\frac{1}{2} + \frac{(1-\alpha)\sigma_x^2}{2\sigma_n^2} + \sqrt{\frac{(1-\alpha)^2\sigma_x^4}{4\sigma_n^4} + \frac{(1+\alpha)\sigma_x^2}{2\sigma_n^2} + \frac{1}{4}}. \quad (16)$$

Then, the corresponding capacities $C(\alpha)$ are derived as

$$C(\alpha) \overset{L=1}{=} -\frac{1}{\ln(2)} \exp\left(1/\overline{\mathrm{SNR}}^{\lim}(\alpha)\right) \times \mathrm{Ei}\left(-1/\overline{\mathrm{SNR}}^{\lim}(\alpha)\right) \quad (17)$$

$$C(\alpha) \overset{L\gg 1}{=} \log 2\left(\frac{1}{2} + \frac{(1-\alpha)\sigma_x^2}{2\sigma_n^2} + \sqrt{\frac{(1-\alpha)^2\sigma_x^4}{4\sigma_n^4} + \frac{(1+\alpha)\sigma_x^2}{2\sigma_n^2} + \frac{1}{4}}\right), \quad (18)$$

and integration with respect to α (see Eq. (13)) leads to the desired spectral efficiencies

$$\mathit{\Gamma}_{\mathrm{sys}}^{\lim} \overset{L=1}{=} \int_0^\beta \frac{-\exp\left(1/\overline{\mathrm{SNR}}^{\lim}(\alpha)\right)}{\ln(2)} \mathrm{Ei}\left(-1/\overline{\mathrm{SNR}}^{\lim}(\alpha)\right) \mathrm{d}\alpha \quad (19)$$

$$\mathit{\Gamma}_{\mathrm{sys}}^{\lim} \overset{L\gg 1}{=} \int_0^\beta \log 2\left(\frac{1}{2} + \frac{(1-\alpha)\sigma_x^2}{2\sigma_n^2} + \sqrt{\frac{(1-\alpha)^2\sigma_x^4}{4\sigma_n^4} + \frac{(1+\alpha)\sigma_x^2}{2\sigma_n^2} + \frac{1}{4}}\right) \mathrm{d}\alpha. \quad (20)$$

Before discussing the influence of load β and the number of paths L on the spectral efficiency, let us address the question what the capacity of the optimum CDMA system employing a joint decoder for all users would be.

First, the spectral efficiency of the optimum decoder for a specific choice of the users' spreading sequences in the μth transmission interval is given as

$$\mathit{\Gamma}_{\mathrm{opt}}[\mu] = \frac{1}{N}\log_2\left(\det\left(\boldsymbol{I} + \frac{\sigma_x^2}{\sigma_n^2}\hat{\boldsymbol{S}}[\mu]\boldsymbol{A}^2[\mu](\hat{\boldsymbol{S}}[\mu])^H\right)\right). \quad (21)$$

Next, it can be shown that for this particular choice of $\hat{\boldsymbol{S}}$ and $\boldsymbol{A}$ the spectral efficiency $\mathit{\Gamma}_{\mathrm{opt}}[\mu]$ can be achieved by MMSE–SIC combined with single user decoding (for references see [1]). More simply,

$$\mathit{\Gamma}_{\mathrm{opt}}[\mu] = \frac{1}{N}\sum_{k=1}^{K} C_{\mathrm{MMSE,k}}[\mu]. \quad (22)$$

Now, assuming again $N \to \infty$ while $\beta = K/N =$ const. it turns out that $C_{\mathrm{MMSE,k}}[\mu]$ approaches in probability

$$\lim_{N\to\infty} C_{\mathrm{MMSE,k}}[\mu] = C(\alpha)[\mu] = \log_2\left(1 + \overline{\mathrm{SNR}}^{\lim}(\alpha)\, a_{k,L}^2[\mu]\right), \quad (23)$$

where as before $\alpha = \frac{K-k}{N}$. Thus, we get for $\lim_{N\to\infty} \mathit{\Gamma}_{\mathrm{opt}}[\mu] \triangleq \mathit{\Gamma}_{\mathrm{opt}}^{\lim}[\mu]$

$$\mathit{\Gamma}_{\mathrm{opt}}^{\lim}[\mu] = \int_0^\beta C(\alpha)[\mu]\mathrm{d}\alpha. \quad (24)$$

Finally, averaging over the various transmission intervals μ we solve

$$\mathit{\Gamma}_{\mathrm{opt}}^{\lim} = \mathop{\mathcal{E}}_{\mu}\{\mathit{\Gamma}_{\mathrm{opt}}^{\lim}[\mu]\} = \int_0^\beta C(\alpha)\mathrm{d}\alpha. \quad (25)$$

More explicitly, the spectral efficiency of the optimum decoder can be achieved by MMSE–SIC with single user decoding for time variant channels supposing perfect channel knowledge, too. So, it can be shown that the result given in Eq. (20) is equal to that found in [1] for the AWGN channel assuming equal power users and a closed analytical solution of the integral is provided in [11].

In Fig. 2 the spectral efficiencies versus power efficiency given in terms of the required energy per bit to noise ratio E_b/N_0 for $\beta = 0.5, 1$ with $L = 1, 2, 4$ as well as the analytical solution for $L \gg 1$ are depicted.

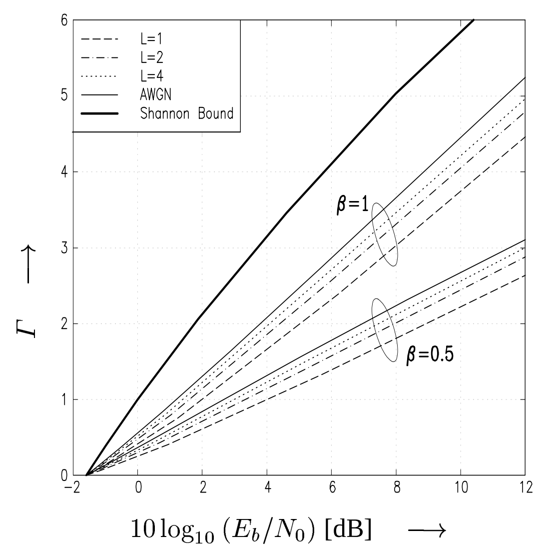

Fig. 2: Spectral Efficiency vs. $10\log_{10}(E_b/N_0)$ for $\beta = 0.5, 1$ with $L = 1, 2, 4, \infty$ and Shannon Bound.

The plot shows that the spectral efficiencies depend strongly on the chosen load $\beta = \frac{K}{N}$. For $\beta = 1$ the spectral efficiency is almost doubled compared to $\beta = 0.5$. In contrast, the influence of the number of propagation paths is quite moderate and as to be expected the corresponding curve for the AWGN channel is approached for rising number of paths L.

Next, let us consider the spectral efficiencies reachable by overloaded systems. In fact, $\mathit{\Gamma}_{\mathrm{sys}}^{\lim}$ vs. $10\log_{10}(E_b/N_0)$ is plotted in Fig. 3 for $L = 1$ and $\beta = 2, 4, 10$. Furthermore, since for $L = 1$ and synchronous transmission it would easily be possible to use an orthogonal transmission scheme the corresponding curve is given, too. The spectral efficiency resulting for an orthogonal scheme is [12]

$$\Gamma_{\mathrm{orth}} = \int_0^\infty \log 2\left(1+\frac{\sigma_x^2}{\sigma_n^2}\zeta\right)\mathrm{e}^{-\zeta}\mathrm{d}\zeta \tag{26}$$

$$= -\frac{1}{\ln(2)}\exp\left(\frac{\sigma_n^2}{\sigma_x^2}\right)\mathrm{Ei}\left(-\frac{\sigma_n^2}{\sigma_x^2}\right). \tag{27}$$

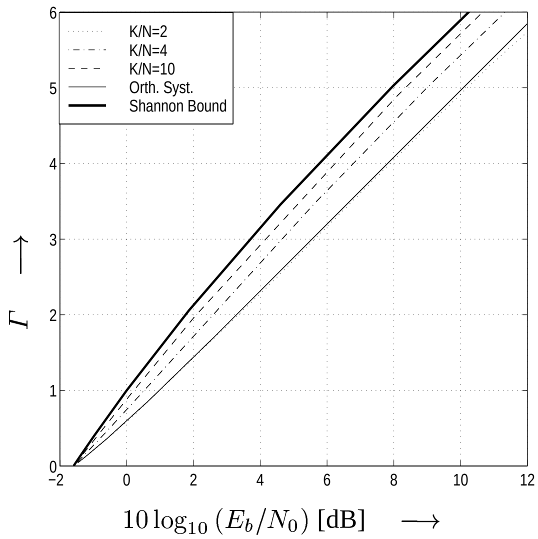

Fig. 3: Spectral Efficiency vs. $10\log_{10}(E_b/N_0)$ for $\beta = 2, 4, 10$ with $L = 1$, orthogonal system and Shannon Bound.

The figure shows that for rising K/N the spectral efficiency of synchronous CDMA employing random spreading sequences for transmission over a flat Rayleigh fading channel converges to the Shannon bound. It has to be emphasized that this is not caused by using the equivalent transmission model, as Eq. (5) represents exactly the actual system for the case $L = 1$ for all values of N. Instead, this result can be explained by the fact that the power allocated to each dimension of the N dimensional space spanned by the spreading sequences reaches an invariant limit for $\beta\to\infty$ being equal for all dimensions. Considering Γ_{orth}, it can be seen that it is only marginally larger than $\Gamma_{\mathrm{sys}}^{\mathrm{lim}}$ with $\beta = 2$. Thus, a relatively small overload suffices to outperform orthogonal transmission.

Finally, in Fig. 4 the spectral efficiency for $\beta = 4$ and $L = 1, 2, 4, 10, \infty$ is depicted.

Studying the curves given in this figure we find again that the increase of $\Gamma_{\mathrm{sys}}^{\mathrm{lim}}$ is very moderate for rising L. Regarding the previous results this stresses again the difference of the two parameters β and L. While $\beta\to\infty$ allows a transmission close to the Shannon bound the gap to the Shannon bound does not vanish even for $L\to\infty$ if β is finite .

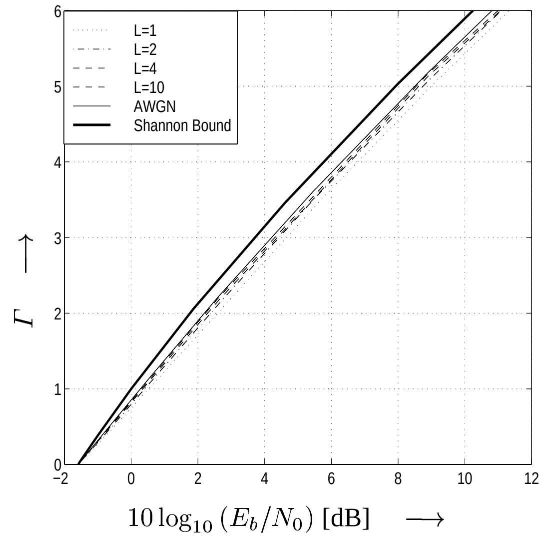

Fig. 4: Spectral Efficiency vs. $10\log_{10}(E_b/N_0)$ for load $\beta = 4$ with $L = 1, 2, 4, 10, \infty$ and Shannon Bound.

IV. Conclusions

In this paper we derived an analytical formula for the spectral efficiency achievable in the limit $N\to\infty$ by synchronous CDMA systems employing MMSE–SIC combined with single user decoding as well as randomly chosen spreading sequences. Equipped with this formula we studied the influence of system load β as well as number of propagation paths L on the system's spectral efficiency. It was shown that the gain resulting from resolving more and more paths supposing an equal gain channel model is quite moderate. In contrast the increase in spectral efficiency due to rising load is significant. So, the results indicate that independently of L the Shannon bound is reached for $\beta\to\infty$. Moreover, we found that for transmission over fading channels and load $\beta > 1$ the spectral efficiency of a nonorthogonal CDMA system with equal power users can exceed that of an orthogonal access scheme. In addition, imposing a limit on the total transmit power the superiority of nonorthogonal CDMA compared to an orthogonal access scheme can be shown, too.

In the same way as above it is also possible to derive the spectral efficiencies of other nonlinear multiuser receivers like the matched filter/decorrelating decision feedback receiver. However, as these receivers are not as good as the MMSE–SIC scheme except for some special cases this has been omitted here.

Finally, it is worth pointing out that the possibility to ignore the intersymbol interference for $N\to\infty$ while $L/N\to 0$ can also be shown by calculating the capacities of a CDMA systems with processing gains N where the first $L-1$ and last $L-1$ received samples in each transmission interval are dumped leading to a lower bound as well as upper bound on the actual system's capacity which merge for $N\to\infty$.

V. References

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