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Probabilistic Signal Shaping in MIMO OFDM via Successive Candidate Generation

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SUMMARY

In this paper, the recently introduced method of channel-coding-based peak-to-average power ratio (PAR) reduction is further developed. On the one hand, not only PAR is considered, but a broad class of signal parameters to be optimized is treated by defining a respective metric to be minimized. On the other hand, the candidate generation and assessment is done successively. If a given tolerable metric limit is met the procedure stops. Analytical expressions for the average number of tested candidates and the achieved performance are derived. Critical metric values, as generalization of the "critical" PAR value $\xi = \log(D)$, where D is the number of carriers, are defined. The theoretical statements are covered by means of numerical simulations using either PAR or cubic metric as desired signal parameter.

1. Introduction

For more than ten years, algorithms for reducing the *peak-to-average power ratio (PAR)* in *orthogonal frequency-division multiplexing (OFDM)* have been discussed in the literature. Recently, a combination of channel coding and selection has been proposed in [6]: processing a number of, say K, OFDM frames jointly (e.g., the parallel signals at an antenna array) Reed–Solomon codes are used to generate further candidate OFDM frames. From them, the K "best" are selected for actual transmission. It has been shown that this scheme is very flexible and it shows very good performance at only moderate complexity.

Besides PAR reduction, other optimization aims on OFDM transmission may be of interest; the aim in future OFDM systems will be to generate signals with certain desired properties. The deliberate generation of signals is usually referred to as *signal shaping* [4]. The approach presented in [6] can easily be generalized to incorporate any desired shaping aim; mathematically, a signal parameter which defines a metric for selection has to be defined and to be used in the selection step.

In this letter, we further generalize the scheme such that encoding and candidate assessment are done successively. If a predefined metric limit is met, i.e., the metric of all selected frames stay below this limit, encoding and candidate assessment stops. Thereby complexity—mainly caused by the calculation of the *inverse discrete Fourier transform ((I)DFT)* and the metric calculation—is reduced significantly. Analytical expressions for the average number of tested candidates and the achieved *complementary cumulative distribution function (ccdf)*, the main performance criterion, are given. Because candidate generation is done randomly, a *probabilistic signal shaping* approach is present.

The letter is organized as follows: In Section 2 the OFDM system model and the performance measure are defined. Section 3 presents the new approach and derives the analytical performance. Results from numerical simulations for PAR and cubic metric reduction given in Section 4. Section 5 concludes the letter.

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2. System Model and Performance Measure

We consider a standard discrete-time OFDM system model using an (I)DFT of length D [2]; all carriers are expected to be active. The frequency-domain OFDM frame is denoted by $\boldsymbol{A} = [A_0, \ldots, A_{D-1}]$; the data symbols A_d, $d = 0, \ldots, D-1$, are drawn from zero-mean $(M = 2^\mu)$-ary QAM or PSK constellations with variance $\sigma_A^2 = \mathrm{E}\{|A_d|^2\}$. The time-domain OFDM frame $\boldsymbol{a} = [a_0, \ldots, a_{DI-1}] = \mathrm{IDFT}_I\{\boldsymbol{A}\}$ is obtained from $\boldsymbol{A}$ via I-times oversampled IDFT, i.e., $a_k = \frac{1}{\sqrt{D}} \sum_{d=0}^{D-1} A_d \cdot \mathrm{e}^{\mathrm{j}2\pi kd/(DI)}$, $k = 0, \ldots, DI-1$. We moreover define the normalized instantaneous power of the time-domain samples as $\alpha_k \stackrel{\text{def}}{=} |a_k|^2/\sigma_A^2$.

Although not exactly true in practice, cf. [9], for analysis it is very convenient to consider the time-domain signal a_k to be complex Gaussian distributed. Thereby, simple analytic expressions can be derived.

Algorithmically we want to control a specific parameter of the transmit signal, we denote it by ξ, and thus generate desired signal properties. Thereby, the probability that this parameter of an OFDM frame exceeds a certain threshold ξ_{th} is of interest, i.e., we consider the complementary cumulative distribution function (ccdf)

$$\mathrm{C}(\xi_{\text{th}}) \stackrel{\text{def}}{=} \Pr\{\xi > \xi_{\text{th}}\} \ . \tag{1}$$

The most important signal parameters to be studied (serving as *selection metric*) are:

Average Transmit Power: A frequent aim in signal shaping [4] is to generate signals with least (normalized) power, i.e., $\xi = \xi_{\text{avg}} \stackrel{\text{def}}{=} \mathrm{E}\{\alpha_k\}$. In OFDM, influencing the signal power is possible if "large" $(M > 4)$ signal constellations are used. In a multi-antenna scheme, sum transmit power is of interest.

Peak-to-Average Power Ratio (PAR): Due to the summation of a large number of statistically independent terms within the IDFT, the time-domain samples a_k tend to be Gaussian and hence exhibit a high PAR $\xi = \xi_{\text{PAR}} \stackrel{\text{def}}{=} \max_k \alpha_k$. The PAR is a well-suited measure if limiting effects (modeled, e.g., by a soft limited) should be characterized.

Cubic Metric: In [10], a cubic metric (CM) has been proposed, which can be reduced to the following form[1] $\xi = \xi_{\text{CM}} \stackrel{\text{def}}{=} \mathrm{E}\{\alpha_k^3\}$. This metric is of particular interest when the power amplifier is assumed to cause non-linear distortions according to the third power of the radio frequency (RF) transmit signal. However, any additional clipping is not included in this metric.

Amplifier-Oriented Distortion Metric: In order to quantify the nonlinear distortions caused by a particular power amplifier, a metric proportional to the energy of the induced error signal quantifying the in- and out-of-band error power) may be of interest.

Subsequently, we consider any of the above parameters ξ (or even any other) as metric for signal design. As we treat multi-antenna transmission where K signals are transmitted in parallel, the *worst-case parameter* (to be minimized) is used as reasonable performance criterion. We assume that the ccdf $\mathrm{C}^{(\text{org})}(\xi_{\text{th}})$ for the original (unshaped) single-antenna scheme and/or the worst case ccdf of the original K-antenna scheme, $\mathrm{C}_{\max}^{(\text{org})}(\xi_{\text{th}})$, are known.

3. Signal Selection Based on Channel Coding with Successive Candidate Generation

In the PAR reduction schemes proposed in [6] K OFDM frames (the antennas of an antenna array or consecutive frames over time) are treated jointly. Given the K initial OFDM frames a collection of N frames is calculated via Reed–Solomon (RS) encoding. Out of these N candidates, a number of (at least) K exhibiting the lowest metric (average power, PAR, CM, ...) are selected for actual transmission. The RS code hence solely serves for candidate generation; the selection step is similar to that in other PAR reduction/metric optimization schemes based on candidate evaluation.

At the receiver, given the K selected and transmitted frames, the initial data has to be recovered. Because RS codes are *maximum distance separable (MDS)*, i.e., they meet the *Singleton bound* on the minimum distance of the code with equality [3], given any selection of K code symbols out of N is sufficient to recover data via erasure decoding.[2]

3.1. Successive Candidate Generation and Stopping Criterion

In [6] it was proposed to assess a fixed number of candidate OFDM frames, which can be done in parallel. In some situations it is advantageous to do this candidate

[1] Scaling and shifting, adapted to the specific 3GPP scenario are eliminated.

[2] The RS decoders have to be informed which K symbols (position within the code word) are non-punctured. Hence, some small amount of side information has to be transmitted as well; we will ignore the required side information in the following.

generation, testing, and selection successively and to stop if some desired performance is already achieved. Thereby, IDFTs and metric calculations (hence complexity) can be saved.

The most obvious stopping condition is to define a tolerable limit ξ_0; if the worst-case metric is below this value, the signal shaping algorithm terminates. Basically, the limit ξ_0 can be chosen arbitrarily. As we will see later on, there exists a range for reasonable choices of ξ_0.

Using the RS code scheme, successive candidate generation and evaluation is very simple. Using, e.g., the well-known feedback shift register systematic encoder for cyclic codes [3] the successive calculation of coded symbols is immediate. Whenever the new candidate has a metric lower than the worst of the K current OFDM frames this new frame is accepted and replaces the worst. Due to the MDS property of the RS codes, no extra caution has to be taken.

3.2. Performance Analysis

3.2.1. Ccdf First we study the ccdf of the successive scheme. Starting point is the ccdf of the original single-antenna OFDM system, $\mathrm{C}^{(\mathrm{org})}(\xi_{\mathrm{th}})$, (without any signal shaping) with D carriers and I-times oversampling.

When a fixed number of $n_{\max}$ candidates is generated the ccdf of the worst-case metric calculates to [6]

$$\mathrm{C}^{(\mathrm{RS})}(\xi_{\mathrm{th}}) = \sum_{l=0}^{K-1} \binom{n_{\max}}{l} \left(1-\mathrm{C}^{(\mathrm{org})}(\xi_{\mathrm{th}})\right)^{l} \cdot (\mathrm{C}^{(\mathrm{org})}(\xi_{\mathrm{th}}))^{n_{\max}-l} \, . \quad (2)$$

Next, assume that potentially an infinite number of candidates can be tested and a metric below the given limit ξ_0 is guaranteed. Then the ccdf reads

$$\mathrm{C}^{(\mathrm{succ},\infty)}(\xi_{\mathrm{th}}) = \Pr\{\max_{\kappa} \xi_\kappa > \xi_{\mathrm{th}} \mid \max_{\kappa} \xi_\kappa \le \xi_0\} = \begin{cases} \frac{\mathrm{C}^{(\mathrm{org})}_{\max}(\xi_{\mathrm{th}}) - \mathrm{C}^{(\mathrm{org})}_{\max}(\xi_0)}{1-\mathrm{C}^{(\mathrm{org})}_{\max}(\xi_0)}, & \xi_{\mathrm{th}} < \xi_0 \\ 0, & \xi_{\mathrm{th}} \ge \xi_0 \end{cases} , \quad (3)$$

where $\mathrm{C}^{(\mathrm{org})}_{\max}(\xi_{\mathrm{th}})$ is the ccdf of the worst-case parameter of the original K-antenna OFDM scheme.

We now restrict the maximum number of trials to $n_{\max}$. If after testing $n_{\max}$ candidates the metric is still above the desired value, for $\xi_{\mathrm{th}} \ge \xi_0$ the ccdf is identical to that of the RS code scheme and for $\xi_{\mathrm{th}} < \xi_0$ it is one. This event occurs with probability $\mathrm{C}^{(\mathrm{RS})}(\xi_0)$. With the complementary probability a metric below the limit is obtained and the ccdf is given by (3). Averaging these two contributions we have for the ccdf of the successive scheme with limitation to $n_{\max}$ candidates

$$\mathrm{C}^{(\mathrm{succ},n_{\max})}(\xi_{\mathrm{th}}) = \begin{cases} \frac{\mathrm{C}^{(\mathrm{org})}_{\max}(\xi_{\mathrm{th}}) - \mathrm{C}^{(\mathrm{org})}_{\max}(\xi_0)}{1-\mathrm{C}^{(\mathrm{org})}_{\max}(\xi_0)} (1-\mathrm{C}^{(\mathrm{RS})}(\xi_0)) + \mathrm{C}^{(\mathrm{RS})}(\xi_0), & \xi_{\mathrm{th}} < \xi_0 \\ \mathrm{C}^{(\mathrm{RS})}(\xi_{\mathrm{th}}), & \xi_{\mathrm{th}} \ge \xi_0 \end{cases} . \quad (4)$$

3.2.2. Average Number of Candidates Next, we assess the average number of candidates to be tested. First, the probability that after testing exactly n candidates the metric threshold is met and the search stops,[3] is given by a negative binomial (or Pascal) distribution [8] (the argument "(ξ_0)" is omitted for convenience; $n = K, K+1, \ldots$)

$$\Pr\{n\} = \binom{n-1}{K-1} (\mathrm{C}^{(\mathrm{org})})^{n-K} (1-\mathrm{C}^{(\mathrm{org})})^{K} \, , \quad (5)$$

Stopping the candidate generation after $n_{\max}$ trials, average complexity (per antenna) is thus given by

$$\bar{n} \stackrel{\mathrm{def}}{=} \frac{1}{K} \left(\sum_{n=K}^{n_{\max}-1} n \cdot \Pr\{n\} + n_{\max} \sum_{n=n_{\max}}^{\infty} \Pr\{n\} \right) . \quad (6)$$

Unfortunately, no analytic expression for this quantity can be given. However, two bounds can be stated: First, $\bar{n} \le n_{\max}$ and, second, average complexity for finite $n_{\max}$ is strictly lower than average complexity in case of $n_{\max} \to \infty$. The later reads, when considering the mean of a negative binomial distribution [8],

$$\bar{n}(\xi_0) = \frac{1}{K} \sum_{n=K}^{\infty} n \cdot \Pr\{n\} = \frac{1}{1-\mathrm{C}^{(\mathrm{org})}} = \frac{1}{\mathrm{cdf}^{(\mathrm{org})}(\xi_0)} \, . \quad (7)$$

Noteworthy, asymptotically the average complexity per antenna is independent of the number of antennas K and is simply given by the reciprocal of the cdf of the respective parameter of the original single-antenna OFDM scheme; this result holds for all oversampling factors I, all modulation alphabets, and all selection metrics. Hence, with the joint operation on K OFDM frames using RS codes it is as arduous to drive the parameter of K signals

[3] K out of the n candidates have lower and $n-K$ larger metric than ξ_0. The probabilities of an OFDM frame to have such a metric are $1-\mathrm{C}^{(\mathrm{org})}(\xi_0)$ and $\mathrm{C}^{(\mathrm{org})}(\xi_0)$, respectively. Since the last candidate has to have a metric lower than ξ_0 (otherwise the search would have stopped earlier), $\binom{n-1}{K-1}$ combinations exist.

simultaneously below the threshold as doing it for one signal.[4]

From the properties of the Pascal distribution [8] we moreover have the variance (per antenna) of the number of candidates

$$\mathrm{var}_n(\xi_0) = \frac{\mathrm{C}^{(\mathrm{org})}(\xi_0)}{K \cdot (1 - \mathrm{C}^{(\mathrm{org})}(\xi_0))^2} \; . \tag{8}$$

From this equation it can be seen that with increasing number of antennas, the number of trials concentrates around its mean. This fact simplifies system design as almost the same number of candidates are to be tested.

3.3. *Critical Parameter Value*

In literature on PAR reduction, e.g., [7, 9], $\xi_{\mathrm{crit}} = \log(D)$ has been identified as a "critical" value,[5] in the sense that there exists schemes which (with high probability) can guarantee that this PAR will not be exceeded. Interestingly, using the popular approximation of Gaussian transmit symbols at Nyquist rate [6] the expected number of candidates reads $\bar{n}_{\mathrm{PAR}}(\xi_0) = (1 - \mathrm{e}^{-\xi_0})^{-D}$ which at the critical value and large number D of carriers evaluates to

$$\bar{n}_{\mathrm{PAR}}(\log(D)) = (1 - 1/D)^{-D} \overset{D\to\infty}{\longrightarrow} \mathrm{e} \; , \tag{9}$$

where $\mathrm{e} = 2.71828\ldots$ is Euler's number. For finite D and non-Gaussian symbols only slight deviations from this number have been observed.

Using the fact, that at the "critical" value, the average number of assessed candidates is e, critical values for all other signal parameters can be defined as well. We hence have the obvious (implicit) definition

$$\mathrm{cdf}^{(\mathrm{org})}(\xi_{\mathrm{crit}}) \overset{!}{=} 1/\mathrm{e} \; , \tag{10}$$

i.e., the ccdf of the parameter in the original single-antenna scheme should be declined to $1 - 1/\mathrm{e} \approx 0.632$. Comparing this behavior to linear circuit theory, in particular the 1st order RC circuit, the critical value is simply the analogon to the time constant.

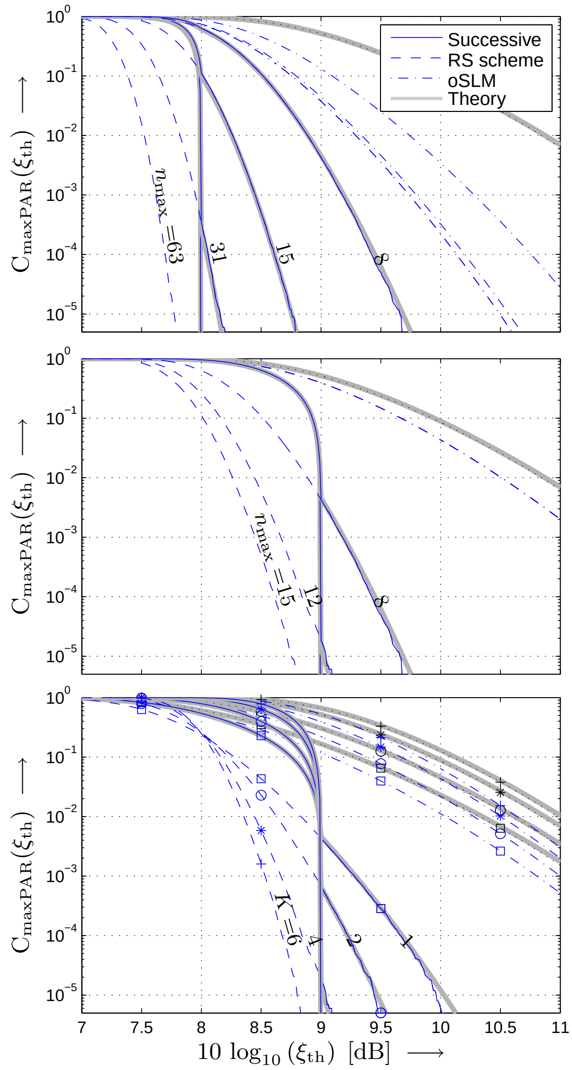

Figure 1. Ccdf of PAR over ξ_{th}. $D = 512$, 4-PSK, $K = 4$, $I = 1$. Solid: proposed successive scheme; dashed: RS coding scheme with a fixed number of candidates equal to $n_{\max}$; dotted: original OFDM; dash-dotted: parallel, independent SLM with the same average number of candidates; gray solid: theoretical curves. Top: $\xi_0 \,\widehat{=}\, 8$ dB; $n_{\max} = 8, 15, 31, 63$. Middle: $\xi_0 \,\widehat{=}\, 9$ dB; $n_{\max} = 8, 12, 15$. Bottom: $K = 1, 2, 4, 6$ with $n_{\max} = 3K$.

4. Numerical Results

The proposed successive scheme is now assessed by means of numerical simulations. Unless otherwise stated an OFDM system with $D = 512$ carriers (all used), 4-PSK modulation per carrier, and $K = 4$ transmit antennas is used. Since the results are representative for all oversampling factors, we mainly concentrate on Nyquist-rate sampling ($I = 1$) and consider the signal parameters PAR and cubic metric.

[4] Most multi-antenna PAR reduction schemes (except that in [5] and the RS coding scheme [6]) do not have this property but required higher complexity.

[5] $\log(\cdot)$ denotes the natural logarithm.

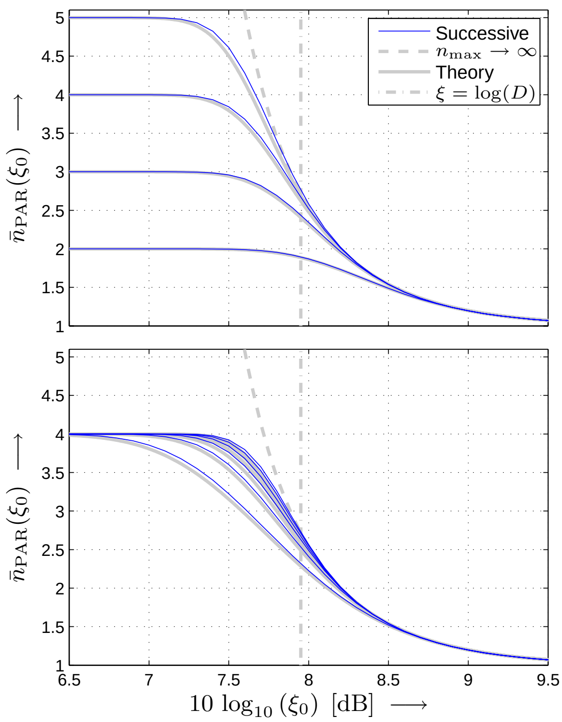

Figure 2. Average number $\bar{n}$ of candidates over the PAR limit ξ_0. $D = 512$, 4-PSK, $I = 1$. Solid: proposed successive RS coding scheme; gray solid: theoretical curves; gray dashed: theoretical curve for $n_{\max} \to \infty$; gray dash-dotted: "critical" PAR value $\xi = \log(D)$. Top: $K = 4$, $n_{\max} = 8, 12, 16, 20$ (bottom to top). Bottom: $n_{\max} = 4, 8, 12, 16, 20, 24$, $K = n_{\max}/4$ (bottom to top).

4.1. Peak-to-Average Power Ratio

In Figure 1 the ccdfs of the worst-case PAR are plotted over the threshold ξ_{th}. Besides the successive scheme (solid), the coding scheme with a fixed number of candidates equal to $n_{\max}$ (dashed), the original OFDM system (dotted), and ordinary (independent, parallel) *selected mapping (SLM)* (dash-dotted) [1] with the number of candidates per antenna equal to the average number of candidates (rounded up to the next integer) of the successive schemes, are shown for reference. Additionally, the theoretical curves are shown in solid gray. The slight differences are due to the used Gaussian approximation.

In the top figure, the PAR limit ξ_0 is fixed to 8 dB, and in the other plots to 9 dB; the "critical" PAR is $10 \log_{10}(\log(512)) = 7.95$ dB. As desired, the ccdf concentrates below the given limit. For smaller $n_{\max}$, a shoulder (simular to the side lobes in a power spectral density) occurs which is given by the ccdf of the coding scheme with the same number of $n_{\max}$ candidates. In each case parallel, independent SLM with the same (average) complexity performs extremely worse.

In the bottom plot of Figure 1, the number of antennas (OFDM frames treated jointly) is varied. In each case the maximum number of candidates is $n_{\max} = 3K$. Although average complexity (per antenna) is the same in all three situations, performance gets much better when increasing K.

In Figure 2 the average number of tested candidates is shown. The theoretical curves for Gaussian signaling are shown in solid gray, and the value of the "critical" PAR is shown via the dash-dotted line.

At the top plot, the number of antennas is fixed to $K = 4$ and the maximum number of assessed candidates in the RS coding scheme is chosen to be $n_{\max} = 8, 12, 16, 20$. For low PAR limits, $\bar{n}$ approaches $n_{\max}/K$; in each case all possible candidates are tested. For high PAR limits ξ_0 all curves merge with the curve for $n_{\max} \to \infty$ and tends to one. Noteworthy, for large $n_{\max}$ at $\xi_0 = \log(D)$ the average number of candidates is $\bar{n} \approx \mathrm{e}$.

If the ratio between K and $n_{\max}$ is fixed (bottom plot), the curves converge at low and high PAR limits. In the middle region, for $K = 1$—which in fact is a single-antenna scheme—the lowest average complexity is observed (but also the worst performance, see Figure 1); for increasing K the curves tend to both bounds; $n_{\max}/K$ and the curve for $n_{\max} \to \infty$.

4.2. Cubic Metric

We now turn to the cubic metric; in Figure 3 the ccdfs of the worst-case CM are plotted over the threshold ξ_{th}. An oversampling factor $I = 4$ is used. The same curves as in the above figures for PAR are given. The ccdf $\mathrm{C}_{\mathrm{CM}}^{(\mathrm{org})}(\xi_{\mathrm{th}})$ for the original (unshaped) single-antenna scheme is approximated from numerical simulations; all other theoretical curves are based thereon.

In the top figure, the CM limit ξ_0 is fixed to 7.5 dB, and in the bottom plot to 8 dB; in the present situation the "critical" CM is approximately 7.55 dB. A comparison with Figure 1 reveals basically the same behavior of CM compared to PAR. All effect and advantages of the successive scheme stated for PAR are equally valid for CM.

In Figure 4 the average number of tested candidates is shown. Again, the same observations as for PAR can be made; the average complexity of e candidates at the "critical" CM value is visible.

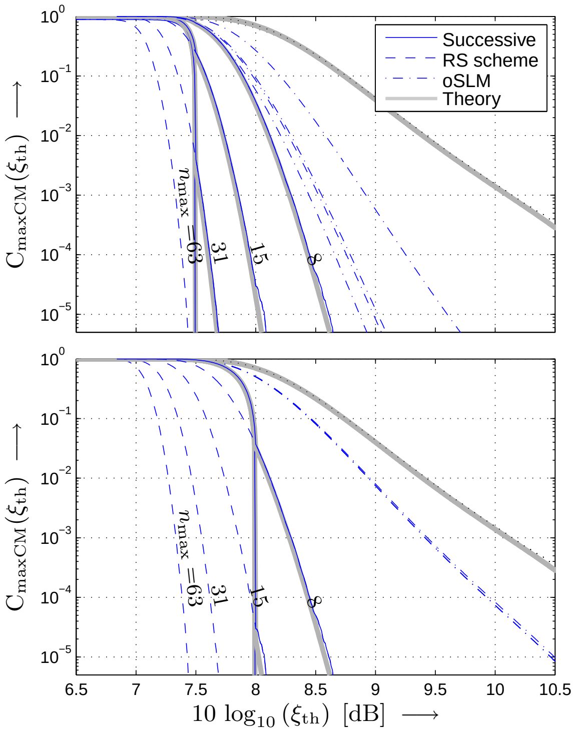

Figure 3. Ccdf of CM over ξ_{th}. $D = 512$, 4-PSK, $K = 4$, $I = 4$. Solid: proposed successive scheme; dashed: RS coding scheme with a fixed number of candidates equal to $n_{\max}$; dotted: original OFDM; dash-dotted: parallel, independent SLM with the same average number of candidates; gray solid: theoretical curves. Top: CM limit $\xi_0 \widehat{=} 7.5$ dB; $n_{\max} = 8, 15, 31, 63$. Bottom: CM limit $\xi_0 \widehat{=} 8$ dB; $n_{\max} = 8, 15, 31, 63$.

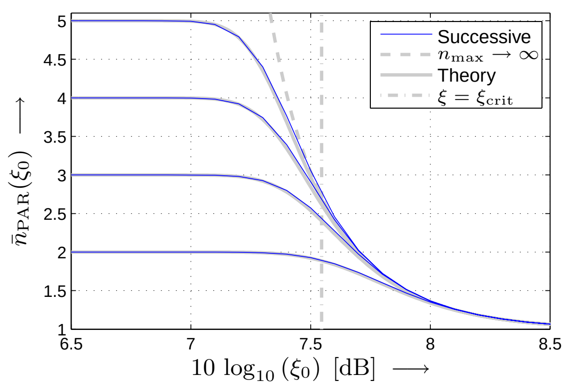

Figure 4. Average number $\bar{n}$ of candidates over the CM limit ξ_0. $D = 512$, 4-PSK, $I = 4$. Solid: proposed successive RS coding scheme; gray solid: theoretical curves; gray dashed: theoretical curve for $n_{\max} \to \infty$; gray dash-dotted: "critical" CM value (here: $\xi = \widehat{=}7.55$ dB) $K = 4$, $n_{\max} = 8, 12, 16, 20$ (bottom to top).

5. Summary and Conclusions

Successive encoding and selection for signal shaping in multi-antenna OFDM has been presented. In particular, Reed–Solomon encoding for the generation of candidates, from which the best are selected (the code is punctured), has been addressed. When stopping the candidate generation after a given metric limit is met, significant reduction in the average number of assessed candidates and hence complexity can be saved.

Interestingly, if the maximally allowed number of candidates, $n_{\max}$, is very large, at the so-called "critical' value (for PAR reduction $\xi_{\mathrm{crit}} = \log(D)$, where D is the number of carriers), the average number of candidates is close to Euler's number e. This number becomes exact for Gaussian Nyquist-rate samples. In general, complexity is simply given by the reciprocal of the cdf of parameter of the original single-antenna OFDM system.

The derived scheme is very flexible: it can be used for all DFT sizes and all modulation alphabets. Similar to channel coding, the shaping capability improves if the number of antennas increases; original OFDM and most other (PAR reduction) schemes deteriorate in this case.

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