Asymptotically Optimal Mappings for BICM with M-PAM and M^2-QAM

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Abstract

We prove the asymptotic optimality of particular binary labeling rules for M-PAM and M^2-QAM constellations in the wideband regime in terms of the so-called parallel-decoding capacity aka. bit-interleaved coded modulation capacity. Shannon's ultimate limit of $-1.59\,\mathrm{dB}$ can be achieved with bit-interleaved coded modulation.

I. Introduction

The analysis of bit-interleaved coded modulation (BICM) [8], [1] is commonly based on an equivalent channel model of several parallel binary input channels [7], [1]. The sum over the individual capacities of these subchannels is generally referred to as *parallel-decoding (PD) capacity* or *BICM capacity*. Depending on the employed binary labeling rule the parallel-decoding capacity exhibits a gap to the coded modulation capacity of the used signal alphabet. In [3] we have shown that for medium to high signal-to-noise ratios (SNR) binary reflected Gray mappings (BRGM) minimize this loss for M-PAM and M^2-QAM; numerical results presented in [4] disproved these characteristics of BRGMs for (very) low SNRs/spectral efficiencies, i.e., in the wideband regime [6]. Following a method developed in [2], which is based on results published in [6], we prove the asymptotic optimality of a particular kind of mappings, i.e., we can show that the ultimate limit of $-1.59\,\mathrm{dB}$ can be achieved with BICM.

II. System Model

For our analysis we concentrate on a detail of the usually employed BICM system model, e.g., [1], namely on mapper and channel. The first maps binary m-tuples $\boldsymbol{x} = [x_1 x_2 \dots x_m] \in \mathbb{F}_2^m$ ($m = \log_2(M)$) onto channel symbols $a \in \mathcal{A} \subseteq \mathbb{C}$ according to a bijective binary labeling rule $\mathcal{M} : \boldsymbol{x} \mapsto a$. The receive symbols y are given as

$$y = ha + n \,. \tag{1}$$

Here, h represents the complex channel coefficient and n the circular symmetric additive white Gaussian noise (AWGN) with variance $\sigma_N^2 = N_0/T_\mathrm{s}$ (N_0: noise power spectral density, T_s: symbol duration). The variance of the channel symbols is denoted by $\sigma_A^2 = E_\mathrm{s}/T_\mathrm{s}$ (E_s: average energy per symbol).[1]

According to [7], the cascade of mapper and channel can be equivalently modeled by m parallel binary input channels, aka. bit levels. Exploiting the chain rule of information theory and the bijectivity of $\mathcal{M}$, the mutual information between the binary input of the μ-th level and the scalar output y reads

$$I_\mathcal{M}(Y;X_\mu) = I(Y;A) - I_\mathcal{M}(Y;X_1 \dots X_{\mu-1} X_{\mu+1} \dots X_m) \,. \tag{2}$$

The mapping-dependent PD capacity as a function in SNR computes to

$$C_\mathcal{M}^\mathrm{PD}(E_\mathrm{s}/N_0) = \sum_{\mu=1}^{m} \max_{p_{X_\mu}(x)} I_\mathcal{M}(Y;X_\mu) \,. \tag{3}$$

[1]Uppercase letters denote the respective random variables; index $\mathcal{M}$ states a dependency on the binary labeling rule.

III. LABELING RULE

We focus on M-PAM and M^2-QAM which can be understood as M-PAM per real dimension. Binary labeling rules defined for an M-PAM can be easily extended to the respective square QAM constellation by independent mapping per quadrature component. The translation of results obtained for an M-PAM to M^2-QAM is then immediate.

A. Set-Partitioning Principle

Definition 1: Consider an M-PAM with $\mathcal{A} = \{\pm 1, \pm 3, \dots, \pm(M-1)\}$ and $\sigma_A^2 = (M^2-1)/3$. Then a mapping according to the set-partitioning principle [5] is given by: partition $\mathcal{A}$ in m steps into $2^m = M$ subsets with a single element each. In the μ-th step partition each of the $k = 2^\mu$ subsets of $\mathcal{A}$ into two subsets with equal cardinalities aiming at a maximized Euclidean distance within the subsets; set the μ-th label bit of the subset with the least element to $b \in \{0, 1\}$, the μ-th label bit of the respective second subset to the complement $\bar{b}$ (for examples see Fig. 1).

We call the result a *strictly regular set-partitioning mapping* as some of the degrees of freedom offered by the initial set-partitioning principle are omitted for the sake of a clearly defined structure.[2]

B. Parallel-Decoding Capacity — Asymptotic Behavior

Proposition 1: Using a mapping as defined in Section III-A BICM asymptotically achieves the ultimate Shannon limit for $E_\mathrm{s}/N_0 \to 0$.

For the proof we use techniques presented in [2], and [6] , which can be briefly sketched as follows: the asymptotic behavior of the parallel-decoding capacity for $E_\mathrm{s}/N_0 \to 0$ is based on its Taylor series expansion

$$C_\mathcal{M}^\mathrm{PD}(E_\mathrm{s}/N_0) = \lambda_1 \cdot (E_\mathrm{s}/N_0) + \lambda_2 \cdot (E_\mathrm{s}/N_0)^2 + o((E_\mathrm{s}/N_0)^2)\,. \tag{4}$$

Rewriting the capacity into a function in E_b/N_0, where E_b denotes the energy per information bit, finally leads to an expression which, among others, depends on (cf. [2, Eq.(9), Appendix B])

$$(E_\mathrm{b}/N_0)_\mathrm{min} = \log(2)/\lambda_1\,, \tag{5}$$

the asymptotic value of E_b/N_0 for $E_\mathrm{s}/N_0 \to 0$. In [2] the authors presented a simple solution for the coefficient λ_1 in BICM scenarios depending on the bit mapping [2, Eq.(15)]

$$\lambda_1 = \frac{1}{2} \cdot \frac{3}{M^2-1} \sum_{\mu=1}^{m} \sum_{b=0}^{1} |\,\mathrm{E}\{A_b^{(\mu)}\}|^2\,. \tag{6}$$

[2]For M^2-QAM the construction principle has to be employed in each dimension independently, resulting in a *per-dimension strictly regular set-partitioning mapping*, cf. [4].

Here $a_b^{(\mu)} \in \mathcal{A}_b^{(\mu)}$ denotes the elements of $\mathcal{A}$ with b as μ-th label bit, cf. Fig. 1, and the factor $3/(M^2-1)$ takes into account, that, in contrast to [2], $\mathcal{A}$ is not normalized to unit variance here.

Regarding the proposed mapping and exploiting symmetries, the addends of the outer sum can be easily given as

$$\frac{1}{2}\sum_{b=0}^{1} |\,\mathrm{E}\{A_b^{(\mu)}\}|^2 = (2^{\mu-1})^2\,. \tag{7}$$

Thus, the coefficient λ_1 can be expressed by a geometric series

$$\begin{aligned} \lambda_1 &= \frac{3}{M^2-1}\sum_{\mu=1}^{m}(2^{\mu-1})^2 = \frac{3}{M^2-1}\cdot\frac{(2^2)^m-1}{2^2-1} \\ &= 1\,. \end{aligned} \tag{8}$$

Using this result, we obtain for the asymptotic E_b/N_0

$$(E_\mathrm{b}/N_0)_\mathrm{min} = \log(2)/1 = \log(2)\,, \tag{9}$$

which coincides with the Shannon limit of $-1.59\,\mathrm{dB}$. For numerical examples we refer to [4].

IV. Conclusion

Based on results published in [6] and [2], we have proven that BICM using M-PAM or M^2-QAM can asymptotically approach Shannon's limit for $E_\mathrm{s}/N_0 \to 0$, if a binary labeling rule as proposed in Section III-A is applied.

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Captions

Fig. 1— Exemplary mappings acc. to definition 1 for 4-PAM and 8-PAM. Subset $\mathcal{A}_1^{(1)}$ marked in gray for 4-PAM; $\mathcal{A}_1^{(2)}$ marked in gray for 8-PAM.

Figure 1

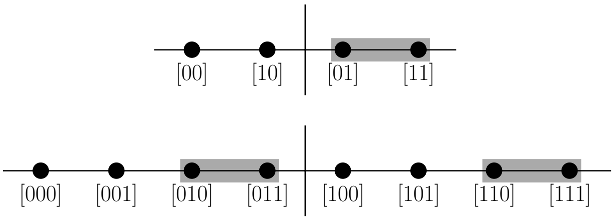

Fig. 1. Exemplary mappings acc. to definition 1 for 4-PAM and 8-PAM. Subset $\mathcal{A}_1^{(1)}$ marked in gray for 4-PAM; $\mathcal{A}_1^{(2)}$ marked in gray for 8-PAM.