A Stopping Radius for the Sphere Decoder and its Application to MSDD of DPSK

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***Abstract*—In this letter we use a lower bound on the packing radius of discrete sets as a stopping radius in the sphere decoder (SD). This enables us to terminate the SD search process as early as possible while preserving the optimality of the final decoder output. As an example we investigate the application of this SD with early termination in multiple-symbol differential detection (MSDD) of differential phase-shift keying (DPSK) transmitted over time-varying fading channels, and provide numerical evidence that the proposed stopping radius achieves a notable complexity reduction.**

I. Introduction

A broad class of detection/decoding problems can be described as a closest-point problem in an L-dimensional discrete set $\boldsymbol{\Lambda} \overset{\text{def}}{=} \{\boldsymbol{\lambda} = \boldsymbol{z}\boldsymbol{G} |\, \boldsymbol{z} \in \mathcal{Z}^L\}$ with generator matrix $\boldsymbol{G}$, and L-dimensional index set $\mathcal{Z}^L$, i.e., to find the member of $\boldsymbol{\Lambda}$ closest to a given $\boldsymbol{x} \in \mathbb{C}^L$:

$$\hat{\boldsymbol{\lambda}} = \underset{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}}{\operatorname{argmin}} \|\boldsymbol{x} - \boldsymbol{\lambda}\|^2 \qquad (1)$$

($\|\cdot\|$ is the Euclidean norm). Most prominent examples include the closest-point problem in lattices ($\mathcal{Z}^L = \mathbb{Z}^L$) [1] and signal detection in multi-antenna systems, cf. e.g. [2]. Furthermore, the shortest-vector problem, i.e., the special case of $\boldsymbol{x} = \boldsymbol{0}$ in (1), is encountered e.g. in maximum-likelihood (ML) multiple-symbol differential detection (MSDD) of differential phase-shift keying (DPSK) [3].

The sphere decoder (SD) according to the Schnorr–Euchner strategy [1] is widely recognized as an efficient algorithm to solve problem (1). In this letter, we present a new stopping rule for terminating the search process of the SD and thus reducing its computational complexity. Our approach is based on a lower bound for the so-called packing radius. Different from other, more aggressive pruning strategies (e.g. [4], [7]), our approach is able to achieve a complexity reduction without sacrificing optimality of the SD output.

This letter is organized as follows. In Section II we first very briefly review the search process of the SD, and then present our approach for early termination. As an application example, in Section III we apply the proposed stopping rule to MSDD of DPSK signals and present numerical results to illustrate the reduction of the SD search complexity. Conclusions are given in Section IV.

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II. Sphere Decoder and Early Termination

A. Sphere Decoder

The SD according to the Schnorr–Euchner strategy first finds the decision-feedback (DF) (or nearest-plane) [1] point by substracting the contribution of already decided dimensions and rounding in every dimension l to the closest $(l-1)$-dimensional hyperplane. The DF point is stored as a preliminary result and its distance to $\boldsymbol{x}$ serves as a search radius. To guarantee optimality, the SD now iteratively checks, if the next-best hyperplanes lie in the search sphere, and updates the preliminary result and search radius, if a new point is found within. If no further points lie inside the search sphere, the optimal solution has been found.

A reasonable measure for the complexity of the SD, which is independent of specific implementation variants, is the number of hyperplanes examined during the search process. This is equivalent to the number of visited nodes in the search tree when formulating the SD as a tree search algorithm [5].

B. Packing Radius

Similar to bounded-minimum distance decoding of algebraic channel codes, the *packing radius* [6]

$$\rho(\boldsymbol{\Lambda}) \overset{\text{def}}{=} \frac{1}{2} \cdot \min_{\boldsymbol{\lambda},\boldsymbol{\lambda}'\in\boldsymbol{\Lambda},\boldsymbol{\lambda}\neq\boldsymbol{\lambda}'} \|\boldsymbol{\lambda}' - \boldsymbol{\lambda}\| \qquad (2)$$

is of central meaning for the decoding process. Spheres of radius equal to the packing radius centered at each point of $\boldsymbol{\Lambda}$ are at most inspheres of the *Voronoi regions*, the set of all points in $\mathbb{C}^L$ that are closer to $\boldsymbol{\lambda}$ than to any other member of $\boldsymbol{\Lambda}$, namely

$$\mathcal{R}_V(\boldsymbol{\lambda}) \overset{\text{def}}{=} \left\{\boldsymbol{x} \in \mathbb{C}^L \,\middle|\, \|\boldsymbol{x} - \boldsymbol{\lambda}\| \le \|\boldsymbol{x} - \boldsymbol{\lambda}'\|, \forall \boldsymbol{\lambda}' \in \boldsymbol{\Lambda}\right\} . \quad (3)$$

See Figure 1 (dotted circles) for an illustration. Consequently, if any examined point of $\boldsymbol{\Lambda}$ lies in the interior of a sphere of stopping radius $R_{\text{stop}} \le \rho(\boldsymbol{\Lambda})$ centered at $\boldsymbol{x}$, i.e., if it fulfills

$$\|\boldsymbol{x} - \boldsymbol{\lambda}\|^2 \le R_{\text{stop}}^2 , \qquad (4)$$

then it is the optimal solution to (1) (see Figure 1, solid circle). Note that the converse does *not* hold.

The stopping radius can be used for early termination of the search process, and thus for reducing the search complexity of the SD. However, $\rho(\boldsymbol{\Lambda})$ is not known for arbitrary $\boldsymbol{\Lambda}$. To find an adequate stopping radius close to the packing radius,

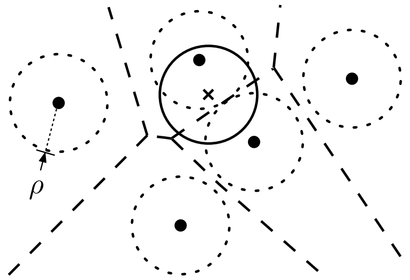

Fig. 1. Sketch of an arbitrary set $\mathbf{\Lambda}$ (black dots) with its Voronoi regions to illustrate the use of the packing radius $\rho(\mathbf{\Lambda})$ for decoding problems.

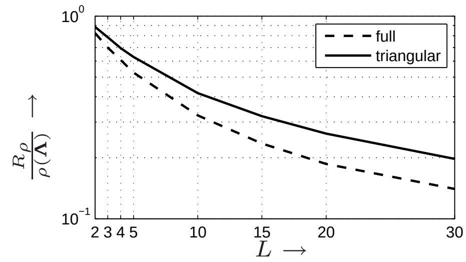

Fig. 2. Average ratio $R_\rho/\rho(\mathbf{\Lambda})$ over lattice dimension L for random generator matrices with zero mean, unit variance, real entries (full and triangular structure).

in the following we derive a lower bound on $\rho(\mathbf{\Lambda})$. A similar, but suboptimal approach has been considered in [7], where the stopping radius is based on the (scaled) expected distance.

We note that our approach is easily combined with different techniques to reduce the SD complexity, such as e.g. optimization of the initial SD search radius.

C. Lower Bound on the Packing Radius

A lower bound on the packing radius follows from the derivation of lattice reduction [8]. We generalize this bound to an arbitrary index set $\mathcal{Z}$ of known minimum squared Euclidean distance

$$d^2_{\min}(\mathcal{Z}) \overset{\text{def}}{=} \min_{z',z\in\mathcal{Z},z'\neq z} |z'-z|^2. \tag{5}$$

Let $\boldsymbol{B}$ be the Gram–Schmidt orthogonal basis of the generator matrix $\boldsymbol{G}$ [9], i.e.,

$$\underbrace{\begin{bmatrix} \boldsymbol{g}_1 \\ \vdots \\ \boldsymbol{g}_L \end{bmatrix}}_{\boldsymbol{G}} = \underbrace{\begin{bmatrix} 1 & & \mathbf{0} \\ & \ddots & \\ \mu_{i,l} & & 1 \end{bmatrix}}_{\boldsymbol{M}} \cdot \underbrace{\begin{bmatrix} \boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_L \end{bmatrix}}_{\boldsymbol{B}} \tag{6}$$

with $\boldsymbol{b}_i\boldsymbol{b}_l^{\mathsf{H}} = 0$ for $i \neq l$. With $\boldsymbol{z},\boldsymbol{z}' \in \mathcal{Z}^L$, $\boldsymbol{z} \neq \boldsymbol{z}'$, the distance of any two points of $\mathbf{\Lambda}$ is lower bounded by

$$\begin{aligned} \|\boldsymbol{\lambda}'-\boldsymbol{\lambda}\|^2 &= \|(\boldsymbol{z}'-\boldsymbol{z})\boldsymbol{M}\boldsymbol{B}\|^2 \\ &= |z'_L - z_L|^2\|\boldsymbol{b}_L\|^2 + \cdots \qquad (7)\\ &\quad + |\mu_{L,1}(z'_L - z_L) + \cdots + (z'_1 - z_1)|^2\,\|\boldsymbol{b}_1\|^2 \\ &\geq d^2_{\min}(\mathcal{Z}) \cdot \min_{i=1,\ldots,L} \|\boldsymbol{b}_i\|^2, \qquad (8) \end{aligned}$$

where we use that $\boldsymbol{z}$ and $\boldsymbol{z}'$ differ at least in one position. Consequently, together with (2) we obtain a lower bound on the packing radius

$$\rho(\mathbf{\Lambda}) \geq R_\rho \overset{\text{def}}{=} \frac{1}{2}\sqrt{d^2_{\min}(\mathcal{Z})} \cdot \min_{i=1,\ldots,L} \|\boldsymbol{b}_i\|. \tag{9}$$

We note that for a generator matrix with triangular structure, (9) simplifies to

$$R_\rho = \frac{1}{2}\sqrt{d^2_{\min}(\mathcal{Z})} \cdot \min_{i=1,\ldots,L} |g_{i,i}|. \tag{10}$$

For lattices we have $\mathcal{Z}^L = \mathbb{Z}^L$, $d^2_{\min}(\mathcal{Z} = \mathbb{Z}) = 1$, and (9) yields the well-known bound from literature [8].

The tightness of the lower bound is illustrated in Figure 2, which shows the average ratio $R_\rho/\rho(\mathbf{\Lambda})$ for lattices with random generator matrices (both full and triangular structure) with zero mean, unit variance, Gaussian distributed real entries. Especially for triangular matrices and moderate L the lower bound is well within a range of the packing radius.

We now set $R^2_{\text{stop}} = \alpha \cdot R^2_\rho$, $\alpha \geq 0$, and obtain an easy to compute stopping radius for the SD. Clearly, $\alpha \leq 1$ does not alter the optimality of the decoder output. The limit $\alpha \rightarrow \infty$ corresponds to DF decoding and the traditional, optimal SD is obtained for $\alpha = 0$ [1]. Hence, this formulation of the stopping radius allows us to include the benchmark cases.

III. MSDD of DPSK Signals

We now illustrate the application of the stopping radius for the SD in MSDD of DPSK transmitted over time-varying Rayleigh fading channels [3], [10].

A. System Model

The sampled received signal can be written as

$$r[i] = h[i]s[i] + n[i]\ , \tag{11}$$

where the transmitted signal $s[i]$ is obtained from differential encoding of information symbols $a[i]$, $s[i] = a[i]s[i-1]$, $s[0] \overset{\text{def}}{=} 1$, and $a[i], s[i] \in \mathcal{M} \overset{\text{def}}{=} \left\{ \mathrm{e}^{\mathrm{j}\frac{2\pi}{M}m} | m = 0, ..., M-1 \right\}$, $n[i]$ is additive white Gaussian noise of one-sided power spectral density N_0, and $h[i]$ denotes the complex channel coefficient. As in [3], [10], we consider Clarke's fading model, i.e., $\mathrm{E}\{h[i]h^*[i-\kappa]\} = J_0(2\pi B_{\mathrm{f}}T\kappa)$, where $J_0(\cdot)$ denotes the zeroth order Bessel function of first kind, and $B_{\mathrm{f}}T$ is the normalized maximum fading bandwidth.

B. Maximum-Likelihood MSDD

It is known that symbol-by-symbol differential detection (DD) suffers an error floor in the bit error rate (BER) curves for $B_{\mathrm{f}}T > 0$. This can be alleviated by ML-MSDD, which performs a joint decision of L information symbols based on a block $\boldsymbol{r} = [r[0], ..., r[L]]$ of $L+1$ receive symbols (w.l.o.g. we consider the block starting at $i = 0$), cf. e.g. [10]. The MSDD decision metric can be written as [3]

$$\hat{\boldsymbol{s}} = \underset{\tilde{\boldsymbol{s}}\in\mathcal{M}^{(L+1)},\tilde{s}_0=1}{\operatorname{argmin}} \|\tilde{\boldsymbol{s}}\boldsymbol{G}\|^2, \tag{12}$$

where $\boldsymbol{G} = (\mathrm{diag}(\boldsymbol{r})\boldsymbol{U})^*$ is an $(L+1)$-dimensional upper-triangular generator matrix, composed of the received signal and the Cholesky factorization of the inverse of the channel matrix $\boldsymbol{C} = \mathrm{E}\{\boldsymbol{h}^{\mathsf{H}}\boldsymbol{h}\} + \frac{N_0}{T}\boldsymbol{I}_{L+1}$, i.e., $\boldsymbol{C}^{-1} = \boldsymbol{U}\boldsymbol{U}^{\mathsf{H}}$.

Due to the triangular structure of the generator matrix, the decision metric can be checked componentwise and (12) can efficiently be solved using the SD (cf. [3] for a detailed description of the algorithm). It can readily be seen that (12) is equivalent to (1) for $\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{\Lambda} = \{\boldsymbol{\lambda} = \tilde{\boldsymbol{s}}\boldsymbol{G} |\, \tilde{\boldsymbol{s}} \in \mathcal{M}^{(L+1)}, \tilde{s}_0 = 1\}$. The differential encoding allows to fix the first component of the trial sequence, and thus the dimensionality of the search problem is again L. Since $d^2_{\min}(\mathcal{Z}) = d^2_{\min}(\mathcal{M}) = 4\sin^2\left(\frac{\pi}{M}\right)$, from (10) we obtain a stopping radius for the SD tailored to MSDD as

$$R^2_{\text{stop}} = |g_{0,0}|^2 + \alpha \cdot \sin^2\left(\frac{\pi}{M}\right) \cdot \min_{i=1,\ldots,L} |g_{i,i}|^2 \,. \qquad (13)$$

The stopping radius is easy to compute and for $\alpha \leq 1$ guarantees the ML property of the SD output.

C. Numerical Results

To illustrate the benefits of early termination with the proposed stopping radius (13), we consider DQPSK ($M = 4$) transmission and $B_\text{f}T = 0.01$ (similar results are obtained for other modulation orders and fading bandwidths).

Figure 3 depicts the average SD complexity $\overline{C}$ with respect to the number of visited hyperplanes during the search process normalized to L over the signal-to-noise ratio (SNR) E_b/N_0 (energy per bit $E_\text{b} = 1/\log_2(M)$) of the SD with stopping radius ($\alpha = 1$, solid lines) in comparison to the SD without stopping radius ($\alpha = 0$, dashed lines) and the benchmark case of DF-DD ($\alpha \to \infty$, dash-dotted line) for several values of L.

We observe that a notable complexity reduction is achieved especially at large SNR. In this regime, the dominant event in the traditional SD search is to find the nearest-plane point and then to check a single alternative in every dimension, which is rejected due to the updated search radius. The former involves the visit of L hyperplanes, while the latter contributes another $L-1$ search steps. Hence, the total search complexity is $2L-1$. With the proposed stopping radius, often the nearest-plane point lies inside the sphere of radius R_stop, and the search terminates after visiting L hyperplanes. Thus, complexity is reduced by approximately a factor of two at no loss of the optimality of the decoder output. Since the nearest-plane point is equal to DF-DD, this leads to ML-MSDD performance at (almost) DF-DD complexity.

Figure 4 visualizes the trade-off (complexity over required SNR to achieve a certain BER) obtained for different parameters α. We observe that the use of the stopping radius with $\alpha = 1$ reduces the complexity to almost its minimum value (L) at still ML performance. Interestingly, a relaxation to $\alpha = 4$ is possible with practically no increase in SNR. This is a result of the underestimation of the packing radius (see Figure 2). We have also included curves (dashed lines) obtained with the fixed stopping radius $R^2_\text{stop} = \beta \cdot L$, $\beta \geq 0$, in Figure 4, which is similar to the stopping criterion in [7]. Although a complexity reduction is also achieved with this approach, error-rate performance deteriorates as β increases. Hence, the proposed, adaptive stopping criterion is clearly superior.

IV. Conclusion

In this letter we have presented a new stopping criterion for the SD, which is based on a lower bound on the packing radius.

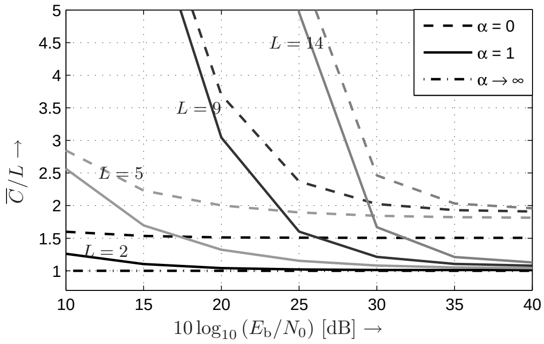

Fig. 3. Average complexity $\overline{C}/L$ over SNR for the SD with stopping radius ($\alpha = 1$) and without stopping radius ($\alpha = 0$) for different L. DF-DD benchmark is also included ($\alpha \to \infty$.)

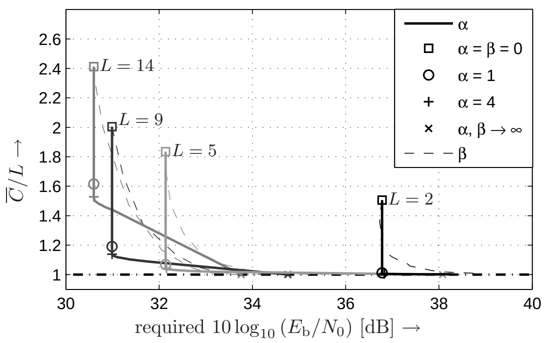

Fig. 4. Average complexity $\overline{C}/L$ of the SD with scaled stopping radius over SNR required to achieve $\text{BER} = 5 \cdot 10^{-4}$, for $M = 4$, $B_\text{f}T = 0.01$, and different L. Curves for a fixed $R^2_\text{stop} = \beta \cdot L$ (dashed lines, cf. [7]) are also included for a comparison.

Its application enables early termination of the search process, and thus reduces the SD complexity. The proposed stopping radius does not alter the final SD output and thus retains its optimality. We have illustrated the application of the stopping radius for the example of MSDD for DPSK using the SD.

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