Asymptotic Analysis of Widely Linear MMSE Multiuser Detection – Complex vs Real Modulation

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Abstract — **In this paper, we give the asymptotic signal–to–interference ratio for the so–called widely linear minimum mean–squared error multiuser receiver for transmission of multiple users to a common receiver with DS–CDMA . Observing that widely linear multiuser detection virtually doubles the spreading factor, we show that transmission with real–valued channel symbols can lead to a higher spectral efficiency than with complex–valued channel symbols when each user employs its own complex–valued random spreading sequence.**

I. Introduction

Recently, closed form solutions for the signal–to–interference ratio (SIR) and spectral efficiencies of various multiuser receivers have been derived assuming random spreading sequences of length $N \to \infty$ while keeping the system load $\beta = K/N$, i.e., the number K of physical users per N chips, fixed [1, 2, 3]. These results are given either for complex–valued random spreading sequences and complex–valued channel symbols or for real–valued random spreading sequences and real–valued channel symbols. Thus, the second order statistics of the received signal is completely described by the covariance matrix of the received signal, which is used for the design of the multiuser interference suppression filters investigated in these papers. Applying those filters which are in this manuscript referred to as *conventional* filters to a transmission scheme with complex–valued random spreading sequences and real–valued channel symbols, it is found that the reachable spectral efficiency is half of that for complex–valued channel symbols assuming equivalent system parameters [4].

A feasible way to increase the SIR (and thereby the spectral efficiency) is to apply so–called *widely linear* multiuser receivers [5, 6, 7]. This possibility arises as a consequence of the fact, that for complete description of the received signal's second order statistics in addition to the covariance matrix also the nonzero pseudocovariance matrix (the covariance matrix of a signal and its complex conjugate [8]) is required. For the sake of clarity we focus in the following on the widely linear MMSE (WLMMSE) filter, but the results can be extended to other widely linear multiuser receivers as well. By now, the gain in the SIR achievable by application of the WLMMSE filter instead of the conventional MMSE filter has mainly been evaluated by means of simulations (see e.g. [7, 6, 5]). Only some special cases like the two users situation could be treated analytically [5].

Here, we give the limiting SIR achievable at the filter output for $N \to \infty, \beta = \text{const}$, and the asymptotic spectral efficiency for the WLMMSE multiuser receiver assuming Gaussian channel symbols. Comparing these results with the corresponding conventional MMSE multiuser receiver, we show that complex–valued signal modulation is not always superior to real–valued signal modulation. For given system load, we derive the points of intersection in the power bandwidth plane where complex–valued and real–valued Gaussian signal constellations yield the same spectral efficiency.

II. Transmission Model

In this paper, the transmission of K users to a common receiver employing DS–CDMA as multiple access technique is considered. For the sake of clarity, we restrict ourselves to a simplified discrete–time equivalent complex baseband transmission model which is mathematically described by

$$\boldsymbol{y}[\mu] \quad = \quad \sum_{k=1}^{K} \boldsymbol{s}_k[\mu] x_k[\mu] + \boldsymbol{n}[\mu] = \boldsymbol{S}[\mu]\boldsymbol{x}[\mu] + \boldsymbol{n}[\mu]. \quad (1)$$

The N dimensional vectors $\boldsymbol{y}[\mu] \triangleq (y_1[\mu], \ldots, y_N[\mu])^T$, and $\boldsymbol{n}[\mu] \triangleq (n_1[\mu], \ldots, n_N[\mu])^T$ designate the received signal and the additive channel noise in the μth transmission interval, respectively.[1] The independent and identically distributed (i.i.d.) samples $n_c[\mu], 1 \le c \le N$, are zero mean rotationally invariant complex–valued Gaussian random variables with variance σ_n^2. The kth user's channel symbols $x_k[\mu]$ are chosen randomly from the set $\mathcal{X}$ with zero mean and variance $\mathcal{E}\{|x_k[\mu]|^2\} = \sigma_x^2$.[2] The elements $s_{ck}[\mu]$ of the spreading sequence $\boldsymbol{s}_k[\mu] \triangleq (s_{1k}[\mu], \ldots, s_{Nk}[\mu])^T$ are zero mean complex–valued random variables with variance $1/N$. For a compact representation we use $\boldsymbol{x}[\mu] \triangleq (x_1[\mu], \ldots, x_K[\mu])^T$, and $\boldsymbol{S}[\mu] \triangleq (\boldsymbol{s}_1[\mu], \ldots, \boldsymbol{s}_k[\mu])$.

[1] $\boldsymbol{x}^T$ denotes the transpose of $\boldsymbol{x}$.

[2] $\mathcal{E}\{|x|^2\}$ designates the expectation of the squared absolute value of x.

III. Asymptotic SIR of MMSE Receivers

In this section, first we review the asymptotic SIR of conventional MMSE interference suppression. Then, we present the asymptotic SIR reachable by widely linear MMSE interference suppression.

A Asymptotic SIR of Conventional Linear MMSE Receiver

The conventional multiuser MMSE filter $\boldsymbol{h}_k[\mu]$ for user k in time slot μ is designed according to

$$\boldsymbol{h}_{k,\mathrm{MMSE}}[\mu] \quad = \quad \underset{\hat{\boldsymbol{h}}_k[\mu]}{\operatorname{argmin}} \; \mathcal{E}\{|x_k[\mu] - \hat{\boldsymbol{h}}_k^T[\mu]\boldsymbol{y}[\mu]|^2\}. \quad (2)$$

Assuming a rotationally invariant complex–valued signal set $\mathcal{X}_{\mathbb{C}}$ and denoting the SIR at the output of $\boldsymbol{h}_{k,\mathrm{MMSE}}[\mu]$ as $\mathrm{SIR}_{k,\mathrm{MMSE},\mathbb{C}}[\mu]$, it was shown in [1] that for $N \to \infty$ and $\beta = K/N = \mathrm{const}$ holds

$$\lim_{K=\beta N \to \infty} \mathrm{SIR}_{k,\mathrm{MMSE},\mathbb{C}} \quad = \quad \mathrm{SIR}_{\mathrm{MMSE},\mathbb{C}} \qquad (3)$$

with $\mathrm{SIR}_{\mathrm{MMSE},\mathbb{C}} = \sqrt{\frac{(1-\beta)^2\mathrm{SNR}^2}{4} + \frac{(1+\beta)\mathrm{SNR}}{2} + \frac{1}{4}} + \frac{(1-\beta)\mathrm{SNR}}{2} - \frac{1}{2}$. Since for real–valued signal sets $\mathcal{X}_{\mathbb{R}}$ with variance σ_x^2 and $\boldsymbol{h}_{k,\mathrm{MMSE}}[\mu]$ only the noise power of one quadrature component is active, the corresponding asymptotic SIR reads

$$\lim_{K=\beta N \to \infty} \mathrm{SIR}_{k,\mathrm{MMSE},\mathbb{R}} = \mathrm{SIR}_{\mathrm{MMSE},\mathbb{R}} = 2\mathrm{SIR}_{\mathrm{MMSE},\mathbb{C}}. \quad (4)$$

B Asymptotic SIR of Widely Linear MMSE Receiver

Applying real–valued channel symbols $x_k[\mu] \in \mathcal{X}_{\mathbb{R}}$, not only the statistical properties described by the covariance matrix of the received signal need to be exploited, but also the information contained in the so–called pseudocovariance matrix $\mathcal{E}\{\boldsymbol{y}[\mu]\boldsymbol{y}^T[\mu]\} = \sigma_x^2 \boldsymbol{S}[\mu]\boldsymbol{S}^T[\mu]$ can be used for interference suppression. In order to account for the correlation between the received vector $\boldsymbol{y}[\mu]$ and the conjugate complex received vector $\boldsymbol{y}^*[\mu]$, the so–called widely linear MMSE filter for user k is designed according to

$$\boldsymbol{h}_{k,\mathrm{WLMMSE}}[\mu] = \underset{\hat{\boldsymbol{h}}_k[\mu]}{\operatorname{argmin}} \; \mathcal{E}\{|x_k[\mu] - \hat{\boldsymbol{h}}_k^T[\mu](\boldsymbol{y}^T[\mu], \boldsymbol{y}^H[\mu])^T|^2\}. \qquad (5)$$

and is solved as [6]

$$\boldsymbol{h}_{k,\mathrm{WLMMSE}}^T[\mu] = \sigma_x^2(\boldsymbol{s}_k^H[\mu], \boldsymbol{s}_k^T[\mu]) \times \left(\sigma_x^2 \begin{pmatrix} \boldsymbol{S}[\mu]\boldsymbol{S}^H[\mu] & \boldsymbol{S}[\mu]\boldsymbol{S}^T[\mu] \\ \boldsymbol{S}^*[\mu]\boldsymbol{S}^H[\mu] & \boldsymbol{S}^*[\mu]\boldsymbol{S}^T[\mu] \end{pmatrix} + \sigma_n^2 \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \right)^{-1}. \qquad (6)$$

Then, the resulting SIR reads

$$\mathrm{SIR}_{k,\mathrm{WLMMSE},\mathbb{R}}[\mu] = \frac{\boldsymbol{h}_{k,\mathrm{WLMMSE}}^T[\mu] \left(\boldsymbol{s}_k^T[\mu], \boldsymbol{s}_k^H[\mu]\right)^T}{1 - \boldsymbol{h}_{k,\mathrm{WLMMSE}}^T[\mu] \left(\boldsymbol{s}_k^T[\mu], \boldsymbol{s}_k^H[\mu]\right)^T}.$$

In [5] it is shown, that $\mathrm{SIR}_{k,\mathrm{WLMMSE},\mathbb{R}}[\mu]$ is the maximum SIR what can be achieved by means of (widely) linear filtering for real–valued channel symbols. In order to derive more general results the asymptotic SIR can be used. It is given in the following theorem.

Theorem 1:

Using real–valued channel symbols which are drawn i.i.d. from the signal set $\mathcal{X}_{\mathbb{R}}$, the signal–to–interference ratio $\mathrm{SIR}_{\mathrm{WLMMSE},\mathbb{R}}(\beta, \mathrm{SNR})$ at the output of the widely linear MMSE filter converges for $N \to \infty$ and $\beta = \mathrm{const}$ to[3]

$$\mathrm{SIR}_{\mathrm{WLMMSE},\mathbb{R}}(\beta, \mathrm{SNR}) = \mathrm{SIR}_{\mathrm{MMSE},\mathbb{C}}(\beta/2, 2\mathrm{SNR}), \quad (7)$$

where $\mathrm{SIR}_{\mathrm{MMSE},\mathbb{C}}(\beta/2, 2\mathrm{SNR})$ denotes the equivalent asymptotic SIR at the output of an MMSE filter if channel symbols from the signal set $\mathcal{X}_{\mathbb{C}} = \{(x_I + \mathrm{j}\, x_Q)/\sqrt{2} | x_I, x_Q \in \mathcal{X}_{\mathbb{R}}\}$ are employed.[4]

It can be shown that this result holds with the appropriate modifications for fading channels with perfect channel state information at the receiver as well. Moreover, regarding the case that the channel state has to be estimated at the receiver and modeling the occuring channel estimation errors as rotationally invariant complex–valued random processes, the resulting asymptotic SIR can be calculated, too.

IV. Comparison of Spectral Efficiencies

Equipped with the asymptotic SIR's we can turn our attention to the spectral efficiencies Γ achievable with complex–valued and real–valued signal modulation for a given power efficiency E_b/N_0 in the asymptotic case. In order to provide a fair comparison we choose Gaussian channel symbols.

Supposing complex–valued channel symbols and a conventional MMSE receiver, the kth user's channel capacity $C_{k,\mathrm{MMSE},\mathbb{C}}$ converges for $N \to \infty, \beta = \mathrm{const}$, to $\lim_{K=\beta N \to \infty} C_{k,\mathrm{MMSE},\mathbb{C}} = C_{\mathrm{MMSE},\mathbb{C}} = \log_2(1 + \mathrm{SIR}_{\mathrm{MMSE},\mathbb{C}})$. Using the definition of a DS–CDMA system's spectral efficiency which reads $\Gamma = \beta C$, for $N \to \infty, \beta = \mathrm{const}$, and the definition of the power efficiency $E_b/N_0 = \mathrm{SNR} \cdot \beta/\Gamma$,[2], the relation between the system's spectral efficiency $\Gamma_{\mathrm{MMSE},\mathbb{C}} = \beta C_{\mathrm{MMSE},\mathbb{C}}$ and its power efficiency $E_b/N_0{}_{\mathrm{MMSE},\mathbb{C}} = \mathrm{SNR}\beta/\Gamma_{\mathrm{MMSE},\mathbb{C}}$ is [3]

$$\frac{E_b}{N_0}_{\mathrm{MMSE},\mathbb{C}} = \frac{2^{\Gamma_{\mathrm{MMSE},\mathbb{C}}/\beta} - 1}{\Gamma_{\mathrm{MMSE},\mathbb{C}}/\beta} \times \frac{2^{\Gamma_{\mathrm{MMSE},\mathbb{C}}/\beta}}{1 + \left(2^{\Gamma_{\mathrm{MMSE},\mathbb{C}}/\beta} - 1\right)(1-\beta)}. \qquad (8)$$

Choosing real–valued Gaussian channel symbols instead of complex–valued one and using a WLMMSE filter at the receiver for interference suppression, the asymptotic single user capacity is $\lim_{K=\beta N \to \infty} C_{k,\mathrm{WLMMSE},\mathbb{R}} =$

[3] The notation $A(b,c)$ denotes the dependence of A on the variables b, c and is written as A for the sake of readability where possible.

[4] The subscripts I, Q denote the inphase and quadrature component, respectively.

$C_{\mathrm{WLMMSE},\mathbb{R}} = 0.5 \cdot \log_2\left(1 + \mathrm{SIR}_{\mathrm{WLMMSE},\mathbb{C}}\right)$. The power efficiency required to achive a desired spectral efficiency Γ is solved as

$$\frac{E_b}{N_0}_{\mathrm{WLMMSE},\mathbb{R}}(\Gamma, \beta) = \frac{E_b}{N_0}_{\mathrm{MMSE},\mathbb{C}}(\Gamma, \beta/2). \quad (9)$$

For illustration, in Fig. (1) the spectral efficiencies $\Gamma_{\mathrm{MMSE},\mathbb{C}}$ and $\Gamma_{\mathrm{WLMMSE},\mathbb{R}}$ vs. $10\log_{10}(E_b/N_0)$ are depicted for various system loads β.

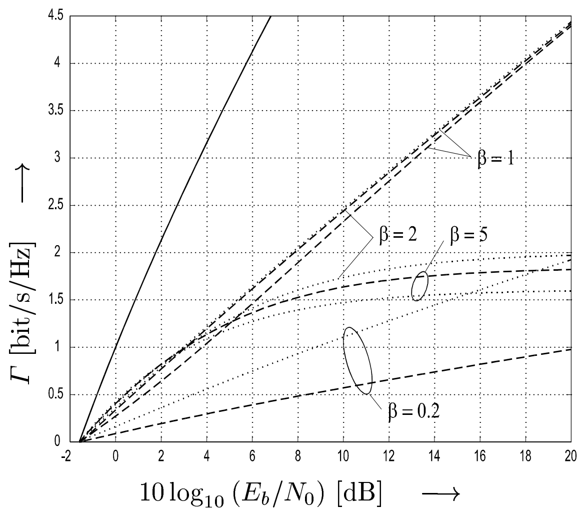

Figure 1: $10\log_{10}(E_b/N_0)$ vs. $\Gamma_{\mathrm{MMSE},\mathbb{C}}$ $(\cdots)$ and $\Gamma_{\mathrm{WLMMSE},\mathbb{R}}$ (- -) for $\beta = 0.2, 1, 2, 5$ and Shannon Bound $(-)$.

It can be seen from this figure that the application of real–valued instead of complex–valued signal modulation seems to provide no benefits in terms of spectral efficiency for low system loads. In contrast to this, for high system loads a considerable gain can be reached. This is a consequence of the (virtually) halved load for real–valued signal modulation and is summarized in the following theorem.

Theorem 2:

For the relation of the spectral efficiencies $\Gamma_{\mathrm{MMSE},\mathbb{C}}$ and $\Gamma_{\mathrm{WLMMSE},\mathbb{R}}$ holds for equal power efficiency E_b/N_0 and equal system load β

$$\Gamma_{\mathrm{MMSE},\mathbb{C}} \begin{cases} > \Gamma_{\mathrm{WLMMSE},\mathbb{R}} & \text{for } \beta < 1 \\ \geq \Gamma_{\mathrm{WLMMSE},\mathbb{R}} & \text{for } \beta = 1, \text{ with} \\ & \text{equality for } \Gamma_{\mathrm{MMSE},\mathbb{C}} \to \infty \\ > \Gamma_{\mathrm{WLMMSE},\mathbb{R}} & \text{for } \beta > 1,\ \Gamma_{\mathrm{MMSE},\mathbb{C}} < \Gamma_{\mathrm{id}} \\ \leq \Gamma_{\mathrm{WLMMSE},\mathbb{R}} & \text{for } \beta > 1,\ \Gamma_{\mathrm{MMSE},\mathbb{C}} \geq \Gamma_{\mathrm{id}} \end{cases}$$

where $\Gamma_{\mathrm{id}} = \beta/2 \cdot \log_2\left(\frac{\beta}{\beta-1}\right)$.

Note if the system load were not defined as $\beta = K/N$ (due to practical reasons) but if it were defined as product of K/N multiplied with the number of real dimensions of the users' signal constellation $\mathcal{X}$, Eq. (9) shows that the performance of both systems would be identical for equal system load. Further, the performance gain for real–valued modulation compared to complex–valued signal modulation is obviously not achievable with the conventional MMSE filter since $C_{\mathrm{MMSE},\mathbb{R}} = 0.5\log_2\left(1 + 2\mathrm{SIR}_{\mathrm{MMSE},\mathbb{C}}\right) \leq C_{\mathrm{MMSE},\mathbb{C}}$.

V. Conclusions

Using that the transmission with real–valued channel symbols and complex–valued spreading sequences of length N is equivalent to transmission with real–valued spreading sequences of length $2N$, we have derived the asymptotic SIR at the output of the widely linear MMSE filter. Based on this, we have shown that the choice of the modulation scheme depends considerably on the number of users to be supported. Applying linear interference suppression according to the MMSE criterion at the receiver, for the given system model it is preferable to use complex–valued modulation schemes for underloaded systems, i.e., $\beta \leq 1$, and to switch to real–valued signal constellations if the system load exceeds 1 and the desired spectral efficiency is larger than the corresponding value for Γ_{id}. This possibility is offered by the application of widely linear filters which can use a reduction in the number of (real) channel symbol dimensions for a better interference suppression.

Considering the strong similarities between DS–CDMA and systems with multiple transmit and receive antennas, it turns out that space–time block codes use inherently the same principle. More explicitly, by transmitting the same information twice where in the second time slot the channel symbols are complex conjugated, the whole signal constellation becomes rotationally variant. This can be used at the receiver for low complexity channel equalization by means of the appropriate widely linear filter [9].

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